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Ngo Van Long, Antoine Soubeyran

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Existence and Uniqueness of Cournot Equilibrium: A Contraction Mapping Approach[*]

Ngo Van Long[†]*, Antoine Soubeyran*[‡]

Résumé / Abstract

Cet article donne des conditions suffisantes pour l'existence et l'unicité de l'équilibre de Cournot. Notre résultat s'applique aux jeux de Cournot à deux étapes. Dans un premier temps, les firmes manipulent leurs coûts marginaux en choisissant un paramètre. Dans un deuxième temps, les firmes font la concurrence à la Cournot.

We provide sufficient conditions for existence and uniqueness of a Cournot equilibrium. The contraction mapping approach is used. Equilibrium is characterized in terms of marginal costs. The result is useful for applications to two-stage games, where, in the first stage, firms incur costs to manipulate their marginal costs of production.

Mots Clés : Équilibre de Cournot, existence, unicité

Keywords: Cournot equilibrium, existence, uniqueness

JEL: L13, L23

[*] Corresponding Author: Ngo Van Long, CIRANO, 2020 University Street 25th floor, Montréal, Qc, Canada H3A 2A5 Tel: (514) 985-4010 Fax: (514) 985-4039 email: longn@cirano.umontreal.ca

[†] McGill University and CIRANO

[‡] GREQAM and Université de la Méditerranée

1 Introduction

This note provides a set of sufficient conditions for the existence and uniqueness of Cournot equilibrium, for any arbitrary number of firms. Previous proofs of existence and/or uniqueness include Friedman (1977), Szidarovsky and Yakowitz (1977), Nishimura and Friedman (1981), Novshek (1985), Kolstad and Mathieson (1987), Gaudet and Salant (1991).

Our approach is different. We prove existence and uniqueness by applying the contraction mapping theorem to a function involving marginal costs. The advantage of our approach is that the equilibrium is characterized in terms of marginal costs, which can be manipulated by firms in an earlier stage, before the Cournot game takes place. This facilitates the study of a class of two-stage Cournot games. (See Long and Soubeyran (1999) for examples of these games.)

2 Sufficient Conditions

There are m firms. Let $M = \{1, ..., m\}$. Let $q_i \geq 0$ denote firm i's output. Its cost function is $C_i(q_i, \alpha_i)$ where $\alpha_i \geq 0$ is a parameter. (We have in mind applications to two-stage Cournot games, where, in the second stage, firms are Cournot rivals, and in the first stage, firms incur costs to change the parameter α_i. For example, α_i may be firm i's capital stock, which is chosen in stage one.) The inverse demand function is $P(Q)$, and $Q = \sum_{i \in M} q_i$. We make the following assumptions.

A1: There exists some $\bar{Q} > 0$ such that $P(Q) > 0$ for $Q \in [0, \bar{Q})$, and $P(Q) = 0$ for $Q \geq \bar{Q}$.

A2: $P''(Q)$ is continuous, $P(0) = \bar{P} > 0$, $P'(Q) < 0$ for $Q \in [0, \bar{Q})$.

A3: $C_i(q_i, \alpha_i)$ is twice continuously differentiable in q_i, and $\frac{\partial C_i}{\partial q_i} > 0$ for all $q_i \in (0, \bar{Q}]$.

A4: $C_i(0, \alpha_i) = 0$, $\frac{\partial C_i(0,\alpha_i)}{\partial q_i} = 0$, and $\frac{\partial^2 C_i}{\partial q_i^2} > 0$ for all $q_i \in (0, \bar{Q}]$.

A5: $q_i P''(Q) + 2P'(Q) < 0$ for all $Q \in [0, \bar{Q})$, and $q_i \in (0, Q]$.

A5(b): $q_i P''(Q) + P'(Q) < 0$ for all $Q \in [0, \bar{Q})$, and $q_i \in (0, Q]$.

A6: $-P'(Q) > \delta > 0$ for $Q \in [0, \bar{Q})$, and, for all $q_i \in (0, \bar{Q}]$, $\frac{\partial^2 C_i}{\partial q_i^2} < b$ for some $b > 0$.

(Note that A5(b) implies A5, it ensures that reaction functions have a negative slope")

Define the marginal cost of firm i as

$$\theta_i = \frac{\partial C_i(q_i, \alpha_i)}{\partial q_i} \tag{1}$$

Equation (1) yields

$$q_i = \rho_i(\theta_i, \alpha_i) \tag{2}$$

with $\frac{\partial \rho_i}{\partial \theta_i} > 0$ for all $\theta_i \geq 0$. Furthermore, because of A4, $\rho_i(0, \alpha_i) = 0$.

Consider the first order condition for an interior equilibrium

$$\widehat{q}_i P'(\widehat{Q}) + P(\widehat{Q}) = \theta_i \tag{3}$$

where the hat denotes equilibrium values. Summing (3) over all $i \in M$, we obtain

$$\widehat{Q} P'(\widehat{Q}) + mP(\widehat{Q}) = m\theta_M \tag{4}$$

where $\theta_M = (1/m) \sum_{i \in M} \theta_i$.

Define $\psi(Q) = QP'(Q) + mP(Q)$. By A2 and A5, $\psi'(Q) < 0$ for al $Q \in [0, \bar{Q})$. Note that $\psi(0) = m\bar{P} > 0$ and $\psi(\bar{Q}) < 0$. It follows that for all $\theta_M \in [0, \bar{P}]$, there exists a unique $\widehat{Q}(\theta_M) \geq 0$ that satisfies (4), and

$$\frac{\partial \widehat{Q}}{\partial \theta_M} = \frac{m}{\psi'} < 0$$

From (3),

$$\widehat{q}_i = \frac{P(\widehat{Q}) - \theta_i}{[-P'(\widehat{Q})]} \tag{5}$$

Using (5) and (2), we get the equilibrium condition for the marginal cost of firm i, given θ_M and α_i,

$$\rho_i(\theta_i, \alpha_i) = \frac{P(\widehat{Q}(\theta_M)) - \theta_i}{[-P'(\widehat{Q}(\theta_M))]} \tag{6}$$

This equation has a unique solution

$$\widehat{\theta}_i = \gamma_i(\theta_M, \alpha_i) \tag{7}$$

The uniqueness of $\widehat{\theta}_i$, given θ_M and α_i, follows from the following facts: (i) the left-hand side of (6) is increasing in θ_i, and takes the value 0 at $\theta_i = 0$, and (ii) the right-hand side of (6) is strictly decreasing in θ_i, given θ_M and α_i, and takes the positive value $P(\widehat{Q}(\theta_M))/[-P'(\widehat{Q}(\theta_M))]$ at $\theta_i = 0$.

The function $\gamma_i(\theta_M, \alpha_i)$ is continuous for all $\theta_M \in [0, \bar{P}]$. Define the function

$$\Gamma(\theta_M, \boldsymbol{\alpha}) = (1/m) \sum_{i \in M} \gamma_i(\theta_M, \alpha_i) \tag{8}$$

where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m)$. For given $\boldsymbol{\alpha}$, the function $\Gamma(\theta_M, \boldsymbol{\alpha})$ is continuous in θ_M for all $\theta_M \in [0, \bar{P}]$ and maps the set $[0, \bar{P}]$ into itself. Thus, from Kakutani's fixed point theorem (Kakutani, 1941), there exists a fixed point $\widehat{\theta}_M$ that satisfies the equation $\Gamma(\theta_M, \boldsymbol{\alpha}) = \theta_M$. We now show the uniqueness of $\widehat{\theta}_M$ by showing that $\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.

Lemma: $\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.

Proof :

Let us find $\partial\gamma_i/\partial\theta_M$. From (6),

$$\frac{\partial \gamma_i}{\partial \theta_M} = \frac{\Lambda_i}{D_i} \left[\frac{-\partial \widehat{Q}}{\partial \theta_M} \right] \tag{9}$$

where

$$D_i = [-P'(\widehat{Q}(\theta_M))] \frac{\partial \rho_i}{\partial \theta_i} + 1 > \frac{\delta}{b} + 1 = D > 1$$

and

$$\Lambda_i = -\widehat{q}_i P''(\widehat{Q}(\theta_M)) - P'(\widehat{Q}(\theta_M)) > 0$$

Hence

$$0 < \frac{\partial \gamma_i}{\partial \theta_M} < \frac{1}{D} \left[\frac{m\Lambda_i}{-P''(\widehat{Q}(\theta_M))\widehat{Q}(\theta_M) - (m+1)P'(\widehat{Q}(\theta_M))} \right]$$

by A5(b) and A6. Thus

$$0 < \frac{\partial \Gamma}{\partial \theta_M} = \frac{1}{D} \left[\frac{-P''(\widehat{Q}(\theta_M))\widehat{Q}(\theta_M) - mP'(\widehat{Q}(\theta_M))}{-P''(\widehat{Q}(\theta_M))\widehat{Q}(\theta_M) - (m+1)P'(\widehat{Q}(\theta_M))} \right] < \frac{1}{D} < 1$$

This shows that $\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.

From the Lemma and the preceding argument, we can now state:

Proposition: Under assumptions A1 to A6, there exists a unique Cournot equilibrium.

3 Concluding Remarks

Our approach has two distinct advantages. First, the contraction mapping approach facilitates numerical computation of the equilibrium. Second, the Cournot equilibrium is characterized in terms of marginal costs, and this facilitates the study of two-stage Cournot games of cost manipulations, where in the first stage firms manipulate their marginal costs by choosing the parameters α_i $(i = 1, ..., m)$ and incurring a cost of manipulation, $\phi_i(\alpha_i)$.

References

[1] Friedman, James W., 1977, *Oligopoly anf the Theory of Games,* North-Holland, Amsterdam.

[2] Gaudet, Gérard, and Stephen W. Salant, 1991, Uniqueness of Cournot Equilibrium: New Results from Old Methods, *Review of Economic Studies 58*, 399-404.

[3] Kakutani, Shizuo, 1941, A Generalization of Brower's Fixed Point Theorem, *Duke Mathematical Journal* 8, 457-9.

[4] Kolstad, Charles D., and Lars Mathieson, 1986, Necessary and Sufficient Conditions for Uniqueness of a Cournot Equilibrium, *Review of Economic Studies 54*, 681-90.

[5] Long, Ngo Van and Antoine Soubeyran, 1999, Cost Manipulation Games in Oligopoly, with Cost of Manipulating, CIRANO working paper 99s-13, CIRANO, Montréal.

[6] Nishimura, Kazuo, and James Friedman, 1981, Existence of Nash Equilibrium in n Person Games without Quasiconcavity, *International Economic Review* 22, 637-48.

[7] Novshek, W. ,1985, On the Existence of Cournot Equilibrium, *Review of Economic Studies 52, 85-98.*

[8] Szidarovsky, F., and S. Yakowitz, 1977, A New Proof of the Existence and Uniqueness of the Cournot Equilibrium, *International Economic Review* 18, 787-9.

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