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Optimal Trading Mechanisms with Ex Ante Unidentified Traders*

Hu Lu[†], *Jacques Robert*[‡]

Résumé / Abstract

Nous analysons les mécanismes optimaux d'échange dans un contexte où chaque participant possède quelques unités d'un bien à être échangé et pourrait être soit un acheteur, soit un vendeur, dépendant de la réalisation des valorisations qui sont de l'information privée des participants. D'abord, le concept de valeur virtuelle est généralisé aux agents qui ne sont pas ex ante identifiés comme acheteur ou vendeur; contrairement au cas où les agents sont bien identifiés, les valeurs virtuelles des agents dépendent maintenant du mécanisme d'échange et ne sont généralement pas monotones même si la distribution des valorisations est régulière. Nous montrons que les mécanismes optimaux d'échange, qui maximisent l'espérance de profit ou de gains d'échange d'un intermédiaire, sont complètement caractérisés par ces valeurs virtuelles. Le phénomène de discrimination incomplète (bunching), qui est ici spécifique aux agents non identifiés ex ante, va être une caractéristique générale dans les mécanismes optimaux. Nous montrons aussi que la règle de répartition aléatoire par laquelle les égalités sont brisées est maintenant un instrument important dans le design de ces mécanismes.

We analyze optimal trading mechanisms in environments where each trader owns some units of a good to be traded and may be either a seller or a buyer, depending on the realization of privately observed valuations. First, the concept of virtual valuation is extended to ex ante unidentified traders; contrary to the case with identified traders, the traders' virtual valuations now depend on the trading mechanism and are generally not monotonic even if the distribution of valuations is regular. We show that the trading mechanisms that maximize a broker's expected profit or expected total gains from trade are completely characterized by some modified monotonic virtual valuations. Here, the bunching phenomena, which is specific to ex ante unidentified traders, will be a general feature in these mechanisms. We also show that the randomization rule by which ties are broken is now an important instrument in the design of the optimal mechanisms.

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1 Introduction

In this paper, we consider the trading problem for a market composed of traders who own some units of an indivisible good and have private information about their own valuations. In this context, a trader holding some units of the good (but less than his satiated demand level) may be either a seller or a buyer, depending on the realization of the privately observed information; however, his role in exchange cannot be identified prior to trade. The standard double auction is then a special case of our model in which all traders possess one or zero units of the good and have unit-demand, so each trader is well identified as a seller or a buyer prior to trade.

We characterize the profit-maximizing trading mechanism and the ex ante efficient trading mechanism by solving the general optimal trading mechanism that maximizes a weighted sum of the expected total gains from trade and the expected profit for the mechanism designer. First, the concept of virtual valuation (Myerson 1981 and 1984) will be extended to ex ante unidentified traders; contrary to the case with identified traders, the traders' virtual valuations now depend on the allocation rules. Additionally, the monotonicity of a trader's virtual valuation fails in any incentive-compatible trading mechanism even if the distribution of valuations is regular. We show that the optimal trading mechanisms are completely characterized by some modified monotonic virtual valuations: the goods will be assigned to the traders whose modified virtual valuations are highest and ties will be broken by randomizing. Also, the participation constraint may be binding at points other than the highest and/or lowest types, and the bunching phenomenon will be a general feature and is specifically associated with ex ante unidentified traders in these optimal mechanisms. In constructing optimal trading mechanisms, we show that the randomization rule by which ties are broken is an important instrument in the design of these mechanisms. As an illustration, using the technique developed in this paper, we solve explicitly the optimal mechanisms for three-trader cases.

The rest of the paper is organized as follows. In section 2, we first define the formal structure of the multilateral trading problem. We then present a general characterization of all incentive compatible and individually rational mechanisms. In section 3, we show how to construct a profit-maximizing mechanism through an algorithm. In section 4, we extend the result of section 3 to a general class of maximization problems and show that the ex ante efficient mechanism is a special case of this class. In section 5, we solve explicitly for the optimal mechanism for the three-trader example.

2 A Trading Problem with Ex Ante Unidentified Traders

We consider the trading problem for a market composed of n traders indexed by $i \in N = \{1, 2, \cdots, n\}$. Each trader i owns k_i units of an indivisible good to be traded and has private information about a preference parameter v_i which is drawn independently from the same distribution F with support $[\underline{v}, \overline{v}]$ and positive continuous density f. Other traders do not observe a trader's type v but know that it is drawn from F. Throughout, we shall assume that the traders want to hold at most k_0 units of the good and are risk neutral. Since the traders have k_0-unit demand, we assume that $k_i \leq k_0$ for all $i \in N$; that is, no trader is initially endowed with more than what he wants to hold.

A trader of type v has preferences represented by the utility function

$$u_i(q, t, v) = v \min(q, k_0 - k_i) - t$$

where $q \geq -k_i$ is the number of units acquired by the trader and t is total spending on these units. The utility is normalized in order to measure the benefit from net trade and we have $u_i(0, 0, v) = 0$ for all v. Note that v is the trader's reservation price for each unit of the k_0 first units of the good.[1] In this context, ex post efficiency requires that all units of good are assigned to the traders with the highest valuations. Denote the total number of units by $K = \sum_{i=1}^{n} k_i$ and let $K = n_0 k_0 + r$ with n_0, r positive integers and $0 \leq r < k_0$. Formally, the ex post efficient allocation can be defined as[2]

$$q_i(v_i, v_{-i}) = \begin{cases} k_0 - k_i, & \text{if } v_i \text{ is among the } n_0 \text{ highest values} \\ r - k_i, & \text{if } v_i \text{ is the } (n_0+1)^{th} \text{ highest value} \\ -k_i, & \text{otherwise} \end{cases} \tag{1}$$

We consider the direct revelation mechanisms in which traders simul-

[1] An alternative assumption is that each trader has a vector of valuations $v_i = (v_i^1, v_i^2, \cdots, v_i^{k_0})$, where v_i^j represents the trader's valuation of his j^{th} unit of the good. If these valuations are not perfectly correlated, it involves a problem of multidimensional uncertainty which appears to be much more complicated. (see Rochet 1985, and Laffont, Maskin, and Rochet 1987) For simplicity, we restrict ourselves to the one-dimensional case.

[2] Notice that since ties occcur with zero probability, they will not affect the expected quantities and so will be ignored in what follows. In general, the private information about valuations and asymmetric initial endowments may lead to some inefficiency in trade; that is, a procedure that can implement the ex post efficient allocation while satisfying individual rationality and budget balance may not always exist. (see Cramton *et al.* 1987, and Lu 1996)

taneously report their valuations[3] $v = (v_1, v_2, \cdots, v_n)$ and then receive an allocation $q(v) = (q_1(v), \cdots, q_n(v))$ and $t(v) = (t_1(v), \cdots, t_n(v))$, where q_i is the net trade for trader i and t_i is the net money transfer from trader i. We assume that each trader is endowed with enough money that any required transfer is feasible. Also, we require that these allocations balance: $\sum_{i=1}^{n} q_i(v) = 0$ for all $v \in [\underline{v}, \bar{v}]^n$. Since all traders want to hold at most k_0 units, we can assume that $-k_i \leq q_i \leq k_0 - k_i$ for all $i \in N$. The pair of outcome functions $\{q, t\}$ is referred to as a *direct trading mechanism*.

Denote $-i = N \backslash \{i\}$ and let $E_{-i}[\,\cdot\,]$ be the expectation operator with respect to v_{-i}. Then we can define the expected net trade and payment for trader i when he announces v_i as his type

$$Q_i(v_i) = E_{-i}[q_i(v_i, v_{-i})]$$

and

$$T_i(v_i) = E_{-i}[t_i(v_i, v_{-i})]$$

so the trader's expected payoff, when everyone truthfully reports, is given by

$$U_i(v_i) = E_{-i}[u_i(q_i(v_i, v_{-i}), t_i(v_i, v_{-i}), v_i)] = v_i Q_i(v_i) - T_i(v_i)$$

The trading mechanism $\{q, t\}$ is *incentive compatible* if each type of each trader wants to report his private information truthfully when others report truthfully

$$U_i(v_i) \geq v_i Q_i(\hat{v}_i) - T_i(\hat{v}_i), \quad \forall i \in N \quad \forall v_i, \hat{v}_i \in [\underline{v}, \overline{v}] \tag{2}$$

As is well known from the *Revelation Principle*, any Bayesian equilibrium outcome of any conceivable mechanism can be obtained as the equilibrium outcome of a direct incentive compatible mechanism. Thus, when we look for trading mechanisms, there is no loss of generality in restricting our attention to direct incentive compatible mechanisms. The mechanism $\{q, t\}$ is interim *individually rational* if all types of all traders are better off participating in the mechanism (in terms of their expected payoff) than holding their initial endowments

$$U_i(v_i) \geq 0, \quad \forall i \in N, \ \forall v_i \in [\underline{v}, \overline{v}] \tag{3}$$

The mechanism $\{q, t\}$ is *feasible* if it is incentive compatible and individually rational. The following lemma develops a necessary and sufficient condition for a mechanism to be feasible.

[3] We assume that the initial endowments $(k_1, k_2, \ldots, k_n)$ are common knowledge, but it is not essential. If the total number of units of the good K is known, we can ask the traders to report their number of units and then forbid trade if the total number reported does not equal K, and implement the mechanism if the reports agree with K.

Lemma 1. *A trading mechanism* $\{q,t\}$ *is feasible if and only if for every* $i \in N$, $Q_i(v_i)$ *is non-decreasing and*

$$U_i(v_i) = U_i(v_i^*) + \int_{v_i^*}^{v_i} Q_i(u)\,du, \quad \forall v_i \in [\underline{v}, \overline{v}] \tag{4}$$

$$U_i(v_i^*) \geq 0, \quad \forall i \in N \tag{5}$$

where

$$v_i^* \in V_i^*(q_i) = \{v_i | Q_i(u) \leq 0, \forall u < v_i; Q_i(w) \geq 0, \forall w > v_i\} \tag{6}$$

Proof: See Appendix

Given a feasible mechanism $\{q,t\}$, since $Q_i(v_i)$ is non-decreasing, $V_i^*(q_i)$ is well-defined and equation (4) implies that expected net utility $U_i(v_i)$ is continuous and convex in v_i. Also, from (6), $U_i(v_i)$ is minimized at $v_i^* \in V_i^*(q_i)$, which will be called the *worst-off* type of the trader. Hence, (5) is equivalent to the individual rationality (3). If $V_i^*(q_i)$ is not a singleton, it is easy to check that $V_i^*(q_i)$ is a closed interval and all worst-off types in $V_i^*(q_i)$ satisfy $Q_i(v_i^*) = 0$. That is, the worst-off types expect to have net trade zero or receive a quantity equal to their initial endowment. Intuitively, a worst-off type expects on average to be neither a buyer nor a seller of the good, and therefore he has no incentive to overstate or understate his valuation. Hence, he does not need to be compensated in order to induce him to report his valuation truthfully, which is why he is the worst-off type of trader. In general, it is no longer clear who is selling and who is buying prior to revelation of types; but on average, trader i is a buyer if his type $v_i > \max V_i^*(q_i)$ and a seller if his type $v_i < \min V_i^*(q_i)$.

Let us define, for all $v \in [\underline{v}, \overline{v}]$

$$\alpha(v) = v - \frac{1 - F(v)}{f(v)} \quad \text{and} \quad \beta(v) = v + \frac{F(v)}{f(v)}$$

$\alpha(\cdot)$ and $\beta(\cdot)$ are referred to as the *virtual valuation* of "buyer-type" and "seller-type" respectively. For any $v_i^* \in [\underline{v}, \overline{v}]$, let

$$\eta(v_i|v_i^*) = \begin{cases} \beta(v_i), & v_i < v_i^* \\ \alpha(v_i), & v_i > v_i^* \end{cases} \tag{7}$$

$\eta(v_i|v_i^*)$ is referred to as the v_i^*-virtual valuation.[4] Given a feasible trading mechanism $\{q,t\}$, if v_i^* is a worst-off type of trader i, $\eta(v_i|v_i^*)$

[4] There is no importance of the value of $\eta(v_i|v_i^*)$ at v_i^* since the probability of $v_i = v_i^*$ is zero.

will be called the trader's virtual valuation of this mechanism. Now, the trader's virtual valuation $\eta(v_i|v_i^*)$ is of "buyer-type" ("seller-type") if and only if his worst-off type is $v_i^* = \underline{v}$ ($v_i^* = \overline{v}$). Typically, a trader may have virtual valuations of both "buyer-type" and "seller-type". For example, when his valuation is greater than v_i^*, he is considered as a buyer since he will receive on average more than his initial endowment; when $v_i < v_i^*$, his virtual valuation is calculated as a seller's since he will receive on average less than his initial endowment. Moreover, $\beta(v) > \alpha(v)$, the trader's virtual valuation is discontinuous at v_i^*, where the trader is expected to change his behavior. Lemma 1 leads to the following characterization of expected revenue from trade of the mechanism.

Lemma 2. *For any function $q = (q_1, \cdots, q_n)$ such that $Q_i(v_i)$ is non-decreasing in v_i for all $i \in N$, there exists a payment function t such that $\{q, t\}$ is incentive compatible and individually rational. The maximum expected revenue from any feasible trading mechanism implementing q is given by*[5]

$$R(q) = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)q_i(v)] = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)Q_i(v_i)] \tag{8}$$

where $v_i^* \in V_i^*(q_i)$ is a worst-off type of trader i for which the individual rationality is binding, i.e., $U_i(v_i^*) = 0$.

Proof : See Appendix

Note that when $V_i^*(q_i)$ is an interval, since the expected net trade $Q_i(v_i)$ is zero on this interval, the expected revenue (8) does not depend on the choice of $v_i^* \in V_i^*(q_i)$. A direct implication of Lemma 2 is revenue equivalence: the expected revenue from a feasible trading mechanism is completely determined by the allocation functions $q = (q_1, \cdots, q_n)$ and the utility levels of the worst-off type $U_i(v_i^*)$ for all $i \in N$. A feasible trading mechanism can be extremely complicated, Lemma 2 helps simplify, however, the problem of mechanism design by establishing that if q is such that $Q_i(\cdot)$ is non-decreasing for all i, then there exists corresponding payment function $t(\cdot)$ such that $\{q(v), t(v)\}$ is an incentive compatible mechanism in which truth-telling is an equilibrium. Thus, the design of incentive compatible and individually rational mechanisms boil down to finding suitable allocation functions $q(\cdot)$.

[5] As a corollary, there exists a trading mechanism that can implement the ex post efficient allocation (1) while satisfying individual rationality and budget balance if and only if the ex post efficient allocation satisfies $\sum_{i=1}^{n} E[\eta(v_i|v_i^*)q_i(v)] \geq 0$.

3 Profit-maximizing Mechanism

We first consider the case where the traders are intermediated by a broker who can be a net source or sink of money, but cannot himself own the objects. In markets in which exchange requires costly search for trading partners, an intermediary can help to reduce the trading frictions. An interesting question is to ask for the mechanism which maximizes the expected profit to the broker, subject to incentive compatibility and individual rationality for traders. That is, if the traders can only trade through the broker, then what is the optimal mechanism for the broker ? From Lemmas 1 and 2, the problem can be written as

$$\mathcal{P}_m \quad \begin{cases} \max \ R(q) = \sum_{i=1}^n E[\eta(v_i|v_i^*)q_i(v)] \\ \text{s.t. } -k_i \leq q_i \leq k_0 - k_i \text{ for all } i \text{ and } \sum_{i=1}^n q_i = 0 \\ Q_i(v_i) \text{ is non-decreasing in } v_i \text{ and } v_i^* \in V_i^*(q_i) \end{cases}$$

In $\mathcal{P}_m$, the virtual valuations $\eta(v_i|v_i^*)$ now depend on the traders' worst-off types, so generally they will depend on the quantity schedules q. Thus, the main difficulty of the problem is that we must consistently determine the traders' worst-off types at which the individual rationality is binding as well as the allocation rules and, at the same time, maximize the expected profit to the broker. Fortunately, we can show that the solution to $\mathcal{P}_m$ can be characterized by some non-decreasing function modified from $\eta(v_i|v_i^*)$ with some appropriate value v_i^*. Here, the bunching phenomena will be a general feature in the optimal mechanism even if the distribution of valuations is regular. To simplify matters, we will assume that $\alpha(\cdot)$ and $\beta(\cdot)$ are continuous and strictly increasing on $[\underline{v}, \bar{v}]$ (the monotone hazard rate assumption) and concentrate our attention on the more interesting type of bunching that is specific to intermediate traders.[6] We can now state and prove the main result of this paper.

Theorem 1. *(i) There exists a unique* $x^* = (x_1^*, \cdots, x_n^*) \in [\underline{v}, \overline{v}]^n$ *for which there will exist at least one allocation* $q^* = (q_1^*, \cdots, q_n^*)$ *which satisfies*

(A) $q^*(v)$ solves

$$\begin{cases} \max \ \sum_{i=1}^n E[\delta_i(v_i|x_i^*)q_i(v)] \\ \text{s.t. } -k_i \leq q_i \leq k_0 - k_i \text{ for all } i \text{ and } \sum_{i=1}^n q_i = 0 \end{cases}$$

where

$$\delta_i(v_i|x_i^*) = \begin{cases} \eta(v_i|x_i^*), & \text{if } v_i \notin [x_i^*, y_i^*] \\ \beta(x_i^*), & \text{if } v_i \in [x_i^*, y_i^*] \end{cases}$$

[6] See, e.g., Myerson (1981), for details in the case where the distribution of valuations is not regular.

with y_i^* such that $\alpha(y_i^*) = \beta(x_i^*)$ or $y_i^* = \overline{v}$ whenever $\alpha(\overline{v}) \leq \beta(x_i^*)$; and

(B) $\forall i$, $Q_i^*(v_i) = E_{-i}[q_i^*(v_i, v_{-i})] = 0$ for $v_i \in [x_i^*, y_i^*]$.

(ii) An allocation $q^* = (q_1^*, \cdots, q_n^*)$ is a solution to $\mathcal{P}_m$ if and only if it satisfies (A) and (B) for this vector x^*.

Proof: (i) We first show that there exists a pair x^* and q^* which satisfy (A) and (B). To prove existence we proceed by submitting an algorithm to construct the solution.

Given some vector x^*, the allocation $q^*(v)$ solves (A) if and only if

$$q_i^*(v_i, v_{-i}|x^*) = \begin{cases} k_0 - k_i, & \text{if } \delta_i(v_i|x_i^*) > l(v|x^*) \\ \text{randomizing}, & \text{if } \delta_i(v_i|x_i^*) = l(v|x^*) \\ -k_i, & \text{otherwise} \end{cases} \tag{9}$$

where $l(v|x^*)$ for any v is such that the number, M_1, of traders with parameter values v_i for which $\delta_i(v_i|x_i^*) \geq l(v|x^*)$ is at least $n_0 + 1$, and the number, M_2, for which $\delta_i(v_i|x_i^*) > l(v|x^*)$ is at most n_0. The randomization rule by which ties are broken is here irrelevant. We can reindex participants so that $k_1 \leq k_2 \leq \cdots \leq k_n$. Since $\delta_i(u|x_i^*) \geq \delta_j(u|x_j^*)$ for all u when $x_i^* > x_j^*$, from (9), we are clearly looking for a vector x^* such that $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$.

For now, consider the solution to (A) $\tilde{q}(v|x^*)$ (given some x^*) where ties are always broken in favor of those with the highest index. Let $\widetilde{Q}_i(x_i^*) = E_{-i}[\tilde{q}(x_i^*, v_{-i}|x^*)]$ be the expected net trade for the participant indexed i when $v_i = x_i^*$ under this alloaction rule. $\widetilde{Q}_i(x_i^*)$ is well-defined and continuous and strictly increasing.[7]

Lemma 3. *For a given vector x^*, there will exist a q^* satisfying Conditions (A) and (B), if and only if we have: $\widetilde{Q}_i(x_i^*) = 0$ whenever*

[7] Since $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$, we have

$$\begin{aligned} \widetilde{Q}_i(x_i^*) = \; & (k_0 - k_i)\text{Prob}\{\text{there are less than } n_0 \text{ other traders who have either} \\ & \text{valuation greater than } y_i^* \text{ and index lower than } i \text{ or} \\ & \text{valuation greater than } x_i^* \text{ and index higher than } i\} \\ & + (r - k_i)\text{Prob}\{\text{there are exactly } n_0 \text{ other traders who have either} \\ & \text{valuation greater than } y_i^* \text{ and index lower than } i \text{ or} \\ & \text{valuation greater than } x_i^* \text{ and index higher than } i\} \\ & - k_i\text{Prob}\{\text{there are at least } n_0 + 1 \text{ other traders who have either} \\ & \text{valuation greater than } y_i^* \text{ and index lower than } i \text{ or} \\ & \text{valuation greater than } x_i^* \text{ and index higher than } i\} \end{aligned}$$

where y_i^* is defined as in (A) of Theorem 1. Obviously y_i^* is continuous and non-decreasing in x_i^*, hence $\widetilde{Q}_i(x_i^*)$ is continuous and strictly increasing.

$x^*_{i-1} < x^*_i < x^*_{i+1}$; *and whenever* $x^*_{i-1} < x^*_i = x^*_{i+1} = \cdots = x^*_j < x^*_{j+1}$ *we have:* $\forall i \leq j$, $\sum_{l=i}^{m} \widetilde{Q}_l(x^*_l) \leq 0$ *for all* $m < j$ *and* $\sum_{l=i}^{j} \widetilde{Q}_l(x^*_l) = 0$.

Proof: (a) Necessity: Note that the δ_i's increase up to some x^*_i, then is constant between x^*_i and y^*_i and increases afterward. A positive probability of a tie between two traders i and j is possible if and only if $x^*_i = x^*_j$. If so, $\delta_i(v|x^*_i) = \delta_j(v|x^*_j) = \beta(x^*_i)$ for a positive mass of values v. Hence, when $x^*_{i-1} < x^*_i < x^*_{i+1}$, the probability of a tie between i and any other participant when $v_i = x^*_i$ is zero. So for all solutions q^* to (A), we must have $Q^*_i(x_i) = \widetilde{Q}_i(x^*_i)$. Ties will occur only if for a subset of participants, $S = \{i, i+1, \cdots, j\}$, we have $x^*_{i-1} < x^*_i = x^*_{i+1} = \cdots = x^*_j < x^*_{j+1}$. In such a case, we have $\widetilde{Q}_i(x^*_i) \leq Q^*_i(x^*_i)$ for all admissible solutions q^* to (A) because i always loses ties against participants of higher indexes in the solution leading to $\widetilde{Q}_i$. Similarly, $\sum_{l=i}^{m} \widetilde{Q}_l(x^*_l) \leq \sum_{l=i}^{m} Q^*_l(x^*_l)$ for all $m < j$. Further, since the expected net trade for a group of participants is independent of how ties are randomly broken between them and the probability of a tie between any two traders $l \in S$ and $h \notin S$ when $v_l = x^*_l$ is zero, we must have $\sum_{l=i}^{j} \widetilde{Q}_l(x^*_l) = \sum_{l=i}^{j} Q^*_l(x^*_l)$ for all solutions q^* to (A).

From the above argument, there will be a q^* satisfying (A) and (B) for a given x^*, only if for all $i \leq j$, $\sum_{l=i}^{m} \widetilde{Q}_l(x^*_l) \leq 0$ for all $m < j$ and $\sum_{l=i}^{j} \widetilde{Q}_l(x^*_l) = 0$ when $x^*_{i-1} < x^*_i = x^*_{i+1} = \cdots = x^*_j < x^*_{j+1}$.

(b) Sufficiency is more involved. If $x^*_{i-1} < x^*_i < x^*_{i+1}$, then $q^*_i(v|x^*_i)$ is uniquely specified by the rule. But if we have $x^*_{i-1} < x^*_i = x^*_{i+1} = \cdots = x^*_j < x^*_{j+1}$, we must specify a (random) tie breaking rule for participants in the subset $S = \{i, \ldots, j\}$. We need to increase the probability that the low index participants are awarded units in case of a tie to garantee that the expected net trades are equal for all participants in S. One can implement any randomization rule, by randomly assigning a hierarchy rank to each participant in the subset S. Participants assigned to a higher hierarchy rank win in case of a tie with participants of lower index. The random assignment process is constructed in such a way that participants in subset S have the same expected net trade, that is 0. For all m, we must have: $\sum_{l \in S} \alpha^l_m(\widetilde{Q}_l(x^*_l) + k_l) = k_m$, where α^l_m is the probability that participant m is assigned rank l in the hierarchy. To achieve this, one can construct a sequence of at most $(j-i)$ one-by-one random permutations from the initial index to a final hierarchy. Details are left to the reader.[8] □

[8] Let m_1 be the lowest index such that net trade is strictly negative and m_2 be the lowest index with strict positive net trade. We must have $m_1 < m_2$, otherwise we would have $\sum_{l=i}^{m_2} \widetilde{Q}_l(x^*_l) > 0$. Then we assign some probability that participant m_1

In order to complete the proof of Theorem 1(i), we need to prove the existence and uniqueness of the vector x^* satisfying the condition of Lemma 3. Existence can be established by construction. We first look for the (unique) vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$ such that[9]

$$\begin{cases} \widetilde{Q}_1(\tilde{x}_1) = 0 \\ \widetilde{Q}_1(\min(\tilde{x}_1, \tilde{x}_2)) + \widetilde{Q}_2(\tilde{x}_2) = 0 \\ \cdots \\ \sum_{i=1}^{l} \widetilde{Q}_i(\min(\tilde{x}_i, \cdots, \tilde{x}_l)) = 0 \\ \cdots \\ \sum_{i=1}^{n} \widetilde{Q}_i(\min(\tilde{x}_i, \cdots, \tilde{x}_n)) = 0 \end{cases}$$

then we set $x_i^* = \min(\tilde{x}_i, \cdots, \tilde{x}_n)$. One can verify that the vector x^* as constructed above satisfies the conditions in Lemma 3.

Uniqueness follows from the strict monotonicity of $\widetilde{Q}_i$'s. Suppose that x^* and z^* satisfy the system of equations in Lemma 3 and $x_i^* < z_i^*$ for some i. Let $i_1 \leq i \leq i_2$ and $i_3 \leq i \leq i_4$ such that $x_{i_1-1}^* < x_{i_1}^* = \cdots = x_i^* = \cdots = x_{i_2}^* < x_{i_2+1}^*$ and $z_{i_3-1}^* < z_{i_3}^* = \cdots = z_i^* = \cdots = z_{i_4}^* < z_{i_4+1}^*$, then we have

$$0 \leq \sum_{l=i_3}^{i_2} \widetilde{Q}_l(x_l^*) < \sum_{l=i_3}^{i_2} \widetilde{Q}_l(z_l^*) \leq 0$$

which is a contradiction.

(ii) Sufficiency: Assume that x^* and q^* satisfy (A) and (B), then we can show that q^* must be a solution to $\mathcal{P}_m$.

Since q^* solves (A), then $q_i^*(v_i, v_{-i})$ is obviously non-decreasing in v_i, and similarly for $Q_i^*(v_i)$. For all feasible allocation $\hat{q} = (\hat{q}_1, \cdots, \hat{q}_n)$ we have

$$\sum_{i=1}^{n} E[\delta_i(v_i|x_i^*)Q_i^*(v_i)] \geq \sum_{i=1}^{n} E[\delta_i(v_i|x_i^*)\widehat{Q}_i(v_i)] \tag{10}$$

is given index m_2 and vice versa. The probability is chosen such that the expected net trade of either m_1 or m_2 becomes zero. We then recalculate the new expected net trade given this reassignment probability and proceed with the new m_1 and m_2. In each round, we bring the net expected trade of at least one participant to zero, so after a finite number of rounds, the net expected trade of all in S will be zero.

[9] We first find $\tilde{x}_1$ such that $\widetilde{Q}_1(\tilde{x}_1) = 0$. Since $\widetilde{Q}_1(\underline{v}) = -k_1 < 0$ and $\widetilde{Q}_1(\overline{v}) = k_0 - k_1 > 0$, strict monotonicity and continuity of $\widetilde{Q}_1(\cdot)$ implies the existence of a unique solution $\tilde{x}_1$. Given $\tilde{x}_1$, we next find $\tilde{x}_2$ such that $\widetilde{Q}_1(\min(\tilde{x}_1, \tilde{x}_2)) + \widetilde{Q}_2(\tilde{x}_2) = 0$. Again, we have $\widetilde{Q}_1(\underline{v}) + \widetilde{Q}_2(\underline{v}) < 0$ and $\widetilde{Q}_1(\tilde{x}_1) + \widetilde{Q}_2(\overline{v}) > 0$, hence there exists a unique $\tilde{x}_2$ which solves the above problem. We then proceed recursively to find all $\tilde{x}_l$. Note that for all l, $\sum_{i=1}^{l} \widetilde{Q}_i(\cdot)$ is continuous, strictly increasing and $\sum_{i=1}^{l} \widetilde{Q}_i(\underline{v}) < 0$ and $\widetilde{Q}_l(\overline{v}) > 0$.

But note that for all $v_i^* \in [x_i^*, y_i^*]$, we have

$$\sum_{i=1}^{n} E[\delta_i(v_i|x_i^*)Q_i^*(v_i)] = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)Q_i^*(v_i)] \tag{11}$$

this follows immediately from the fact that by construction, we have $\delta_i(v_i|x_i^*) = \eta(v_i|v_i^*)$ for $v_i \notin [x_i^*, y_i^*]$ and $Q_i^*(v_i) = 0$ for $v_i \in [x_i^*, y_i^*]$.

If $\widehat{Q}_i(v_i)$ is non-decreasing, then $V_i^*(\hat{q}_i)$ is well-defined. For any $\hat{v}_i \in V_i^*(\hat{q}_i)$, we have

$$\begin{aligned}\sum_{i=1}^{n} E[\delta_i(v_i|x_i^*)\widehat{Q}_i(v_i)] &= \sum_{i=1}^{n} E[\eta(v_i|\hat{v}_i)\widehat{Q}_i(v_i)] \\ &\quad + \sum_{i=1}^{n} E[(\delta_i(v_i|x_i^*) - \eta(v_i|\hat{v}_i))\widehat{Q}_i(v_i)] \\ &\geq \sum_{i=1}^{n} E[\eta(v_i|\hat{v}_i)\widehat{Q}_i(v_i)] \end{aligned} \tag{12}$$

the inequality follows from the facts that $\delta_i(v_i|x_i^*) - \eta(v_i|\hat{v}_i) \leq 0$ and $\widehat{Q}_i(v_i) \leq 0$ when $v_i < \hat{v}_i$, and $\delta_i(v_i|x_i^*) - \eta(v_i|\hat{v}_i) \geq 0$ and $\widehat{Q}_i(v_i) \geq 0$ when $v_i > \hat{v}_i$. Using (10) and (11), (12) implies

$$\sum_{i=1}^{n} E[\eta(v_i|v_i^*)Q_i^*(v_i)] \geq \sum_{i=1}^{n} E[\eta(v_i|\hat{v}_i)\widehat{Q}_i(v_i)] \tag{13}$$

for all feasible $\hat{q}$ with non-decreasing $\widehat{Q}_i$'s. Hence q^* solves $\mathcal{P}_m$.

Necessity: Now suppose that some alternative solution $\hat{q}$ to $\mathcal{P}_m$ exists. Clearly, (10) cannot hold with strict inequality: if so, (13) will hold with strictly inequality contradicting the assumption that $\hat{q}$ solves $\mathcal{P}_m$. $\hat{q}$ must also solve the program in (A). Now suppose that $\hat{q}$ does not satisfy (B), i.e., for at least one i there is an open set $(u, w) \subset [x_i^*, y_i^*]$ such that for all $v_i \in (u, w)$, $\widehat{Q}_i(v_i) \neq 0$. If so, for some $\hat{v}_i \in V_i^*(\hat{q}_i)$, there will exist a positive probability of values v_i such that $\delta_i(v_i|x_i^*) > \eta(v_i|\hat{v}_i)$ (or $< \eta(v_i|\hat{v}_i)$) and $\widehat{Q}_i(v_i) < 0$ (or $\widehat{Q}_i(v_i) > 0$). Hence

$$E[(\delta_i(v_i|x_i^*) - \eta(v_i|\hat{v}_i))\widehat{Q}_i(v_i)] > 0$$

The inequality in (12) is strict, so must be the inequality in (13), which contradicts the assumption that $\hat{q}$ is a solution to $\mathcal{P}_m$. Therefore, $\hat{q}$ must satisfy (B). □

From Theorem 1, in the optimal allocation, the objects should always be assigned to the traders with the highest modified virtual valuations

$\delta_i(v_i|x_i^*)$. But, since $\delta_i(v_i|x_i^*)$ is constant on $[x_i^*, y_i^*]$, ties may occur with positive probability and should be broken by randomizing. The randomization rule by which ties are broken may now affect the traders' expected quantities and hence becomes an important instrument in the design of the optimal trading mechanism. Intuitively, the optimal allocation is designed in such a way that higher types are expected to be buyers and lower types are expected to be sellers, yielding the most gains from trade. By construction, we have $\underline{v} < x_i^* < \overline{v}$ when $0 < k_i < k_0$, that is, the individual rationality constraint for an intermediate trader is necessarily binding between the highest and lowest types. Also, we have $x_i^* \leq x_j^*$ when $k_i \leq k_j$, but we may also have $x_i^* = x_j^*$ when $k_i < k_j$. That is, two traders with different initial endowments may have the same types at which the individual rationality is binding and they expect to be neither a buyer nor a seller. In general, j does not always win ties against i in the optimal allocation, so if x_j^* is separated from x_i^* (just above x_i^*), the chance of winning of x_j will be increased by a positive probability. This implies that x_i^* and x_j^* can be separated only if $k_j - k_i$ is large enough.

Theorem 1 proves a characterization of the optimal trading mechanism. In the following theorem, we provide some basic comparative results.

Theorem 2. *The expected revenue from the optimal trading mechanism is (i) non-decreasing when the initial endowments are more symmetric and (ii) strictly increasing with k_0.*

Proof: (i) Let q^* be the optimal allocation with initial endowments k_i and k_j. Now suppose that we reallocate the initial endowment so that i receives $k_i^a = ak_i + (1-a)k_j$ and j has $k_j^a = (1-a)k_i + ak_j$, $0 < a \leq \frac{1}{2}$. We first claim that the allocation, where $q_l^a = q_l^*$ for all $l \neq i, j$ and $q_i^a = aq_i^* + (1-a)q_j^*$ and $q_j^a = (1-a)q_i^* + aq_j^*$, is feasible. Indeed, for any v, $\sum_{i=1}^n q_i^a(v) = \sum_{i=1}^n q_i^*(v) = 0$; $-k_i \leq q_i^* \leq k_0 - k_i$ and $-k_j \leq q_j^* \leq k_0 - k_j$ imply $-(ak_i + (1-a)k_j) \leq aq_i^* + (1-a)q_j^* \leq k_0 - ak_i - (1-a)k_j$.

Second, we show that q^a generates at least as much return as q^*. We have

$$\begin{aligned}
&E[\eta(v_i|v_i^*)Q_i^*(v_i)] + E[\eta(v_j|v_j^*)Q_j^*(v_j)] \\
&\quad \leq \quad E[(a\eta(v_i|v_i^a) + (1-a)\eta(v_i|v_j^a))Q_i^*(v_i)] + E[((1-a)\eta(v_j|v_i^a) + a\eta(v_j|v_j^a))Q_j^*(v_j)] \\
&\quad = \quad E[\eta(v_i|v_i^a)(aQ_i^*(v_i) + (1-a)Q_j^*(v_i))] + E[\eta(v_j|v_j^a)(aQ_j^*(v_j) + (1-a)Q_i^*(v_j))] \\
&\quad = \quad E[\eta(v_i|v_i^a)Q_i^a(v_i)] + E[\eta(v_j|v_j^a)Q_j^a(v_j)]
\end{aligned}$$

The first inequality follows from the fact that $(\eta(v_i|v_i^a) - \eta(v_i|v_i^*))Q_i^*(v_i) \geq 0$ for all v_i^a. (A similar result applies for j). Note that that the inequality is strict if the range over which $Q_i^*(v_i) = 0$ and the range over which

$Q_j^*(v_j) = 0$ do not coincide. In this case, there will exist values v_i such that $a\eta(v_i|v_i^a) + (1-a)\eta(v_i|v_j^a) > \eta(v_i|v_i^*)$ and $Q_i^*(v_i) > 0$.

(ii) To show that expected revenue must strictly increase in k_0, notice that the optimal allocation q^* given some initial k_0' is feasible with a higher $k_0'' > k_0'$. Furthermore, q^* is not optimal since q^* cannot satisfy condition (A) of Theorem 1. There must exist a $\hat{q}$ which generates higher return. □

Intuitively, when the initial endowments are more unevenly distributed amongst the participants, then everyone with more (less) good would more likely expect to be a seller (buyer) and want to overstate (understate) his valuation. Such behavior, which is the essence of bargaining, may increase the "bribe" one must offer them to induce truthful revelation of private information. Hence, there will be less expected revenue from the trading mechanism. On the other hand, the optimal allocation requires that all goods go to the traders with the highest valuations. When the traders have a higher level of demand, then the number of traders to whom the goods are assigned is smaller. Hence, less incentives are required to induce truthful revelation of the smaller number of traders with the highest valuations; additionally, the gains from trade are higher. Thus, the expected revenue from the optimal trading mechanism is increasing with k_0.

4 Ex ante efficient mechanisms

We can also extend the result of section 3 by introducing a general objective function that is a weighted sum of the expected total gains from trade and the expected revenue for the mechanism designer, i.e., for any $\lambda \in [0, 1]$

$$W_\lambda(q) = (1-\lambda)E\left[\sum_{i=1}^{n} v_i q_i(v)\right] + \lambda R(q)$$

We seek a mechanism that maximizes the above objective function subject to the incentive compatibility and individual rationality for traders. Using some algebra, $W_\lambda(q)$ can be rewritten as

$$W_\lambda(q) = E\left[\sum_{i=1}^{n} \eta(v_i|v_i^*, \lambda) q_i(v)\right]$$

where for any $v_i^* \in V_i^*(q_i)$ and $\lambda \in [0, 1]$,

$$\eta(v_i|v_i^*, \lambda) = \begin{cases} \alpha(v_i, \lambda) = v_i + \lambda \frac{F(v_i)-1}{f(v_i)}, & v_i > v_i^* \\ \beta(v_i, \lambda) = v_i + \lambda \frac{F(v_i)}{f(v_i)}, & v_i < v_i^* \end{cases}$$

From Lemmas 1 and 2, the maximization problem can be written as

$$\mathcal{P}_\lambda \quad \left\{ \begin{array}{l} \max\ W_\lambda(q) = E[\sum_{i=1}^n \eta(v_i|v_i^*,\lambda)q_i(v)] \\ \text{s.t. } -k_i \le q_i \le k_0 - k_i \text{ for all } i \text{ and } \sum_{i=1}^n q_i = 0 \\ Q_i(v_i) \text{ is non-decreasing in } v_i \text{ and } v_i^* \in V_i^*(q_i) \end{array} \right.$$

Notice that if $\alpha(v_i)$ and $\beta(v_i)$ are strictly increasing, it is straightforward to verify that for any λ, $\alpha(v_i,\lambda)$ and $\beta(v_i,\lambda)$ are also strictly increasing in v_i, so the virtual valuation $\eta(v_i|v_i^*,\lambda)$ is increasing over $[\underline{v}, v_i^*)$ and $(v_i^*, \overline{v}]$, but there is a "buyer-seller" spread of virtual valuation at v_i^*. Also, the trader's virtual valuations are distorted downward (upward) to be below (above) his true reservation values when he expects to be a buyer (seller).

$\mathcal{P}_\lambda$ has exactly the same structure as $\mathcal{P}_m$, so to solve $\mathcal{P}_\lambda$, as in Theorem 1 we should first modify the virtual valuations by defining a monotonic function

$$\delta_i(v_i|x_i^*,\lambda) = \left\{ \begin{array}{ll} \eta(v_i|x_i^*,\lambda), & \text{if } v_i \notin [x_i^*, y_i^*] \\ \beta(x_i^*,\lambda), & \text{if } v_i \in [x_i^*, y_i^*] \end{array} \right.$$

with y_i^* such that $\alpha(y_i^*,\lambda) = \beta(x_i^*,\lambda)$ or $y_i^* = \overline{v}$ whenever $\alpha(\overline{v},\lambda) \le \beta(x_i^*,\lambda)$. Also, let

$$q_i^*(v_i, v_{-i}|x^*,\lambda) = \left\{ \begin{array}{ll} k_0 - k_i, & \text{if } \delta_i(v_i|x_i^*,\lambda) > l(v|x^*,\lambda) \\ \text{randomizing}, & \text{if } \delta_i(v_i|x_i^*,\lambda) = l(v|x^*,\lambda) \\ -k_i, & \text{otherwise} \end{array} \right. \quad (14)$$

where $l(v|x^*,\lambda)$ is similar to $l(v|x^*)$ in definition (9). The following theorem characterizes the optimal mechanism for $\mathcal{P}_\lambda$, and can be obtained as a direct generalization of Theorem 1.

Theorem 3. *(i) There exists a unique $x^* = (x_1^*, \cdots, x_n^*) \in [\underline{v}, \overline{v}]^n$ for which there will exist at least one randomization rule such that the allocation (14) satisfies $Q_i^*(v_i|x^*,\lambda) = E_{-i}[q_i^*(v_i, v_{-i}|x^*,\lambda)] = 0$ for all $v_i \in [x_i^*, y_i^*]$. (ii) An allocation $q^* = (q_1^*, \cdots, q_n^*)$ is a solution to $\mathcal{P}_\lambda$ if and only if it satisfies (14) and $Q_i^*(v_i|x^*,\lambda) = 0$ for all $v_i \in [x_i^*, y_i^*]$.*

Theorem 3 can be useful to characterize the most efficient trading mechanism subject to the constraint that traders are not subsidized. In some economic environments, (e.g., double auctions,) ex post efficiency cannot be achieved by any individually rational mechanism, unless some outsider is willing to provide a subsidy to the traders for participating in the trading mechanism. Since there is no reason to subsidize a private goods market, ex post efficiency is unattainable. Therefore, it is natural to seek a mechanism that maximizes expected total gains from trade, subject to the incentive compatibility and individual rationality constraints, as well as the market-maker's budget constraint. (This maximization is equivalent to maximizing the sum of the traders' ex ante expected utilities because each trader's utility function is separable in money and his valuation.) That is, we are looking for the *ex ante* efficient mechanism[10] which solves

$$\mathcal{P}_s \quad \begin{cases} \max\ E[\sum_{i=1}^n v_i q_i(v)\,] \\ \text{s.t. } -k_i \leq q_i \leq k_0 - k_i \text{ and } \sum_{i=1}^n q_i = 0 \\ \{q,t\} \text{ is feasible and } E[\sum_{i=1}^n t_i(v)] \geq 0 \end{cases}$$

This problem can be rewritten as

$$\mathcal{P}_s \quad \begin{cases} \max\ E[\sum_{i=1}^n v_i q_i(v)\,] \\ \text{s.t. } -k_i \leq q_i \leq k_0 - k_i \text{ and } \sum_{i=1}^n q_i = 0 \\ Q_i(v_i) \text{ is non-decreasing in } v_i \text{ and } E[\sum_{i=1}^n \eta(v_i|v_i^*) q_i(v)] \geq 0, \\ \qquad \text{where } v_i^* \in V_i^*(q_i) \end{cases}$$

We can show that $\mathcal{P}_s$ is a special case of $\mathcal{P}_\lambda$ for some λ. In fact, if the ex post efficient allocation is a solution to problem $\mathcal{P}_s$, then it is also a solution to $\mathcal{P}_\lambda$ for $\lambda = 0$. Otherwise, we need to set $\lambda = \frac{\mu}{1+\mu}$ where μ corresponds to the Lagrangian multiplier associated with the no-subsidy constraint for the market-maker. Since the ex post efficient allocation is not a solution to problem $\mathcal{P}_s$, any allocation that satisfies the no-subsidy constraint with equality and solves $\mathcal{P}_\lambda$ must be a solution to $\mathcal{P}_s$. Thus, any solution q to $\mathcal{P}_\lambda$ for some λ such that

$$E\left[\sum_{i=1}^n \eta(v_i|v_i^*) q_i(v)\right] = 0 \tag{15}$$

must be a solution to $\mathcal{P}_s$.

5 Three-trader Cases

We consider the case where there are 3 traders and everyone possesses k_i units of the good with $k_1 + k_2 + k_3 = K$ and wants to hold at most K units. We normalize $K = 1$, so each k_i corresponds to i's share of total units. We assume that traders' valuations are drawn from a uniform distribution F on $[0, 1]$. We are looking for the profit-maximizing trading mechanism.

[10] See, e.g., Holmström and Myerson 1983. Here we focus just on the ex ante efficient mechanism that places equal welfare weights on every trader and maximizes the sum of all traders' expected gains from trade.

Under the uniform distribution on $[0,1]$, we have $\alpha(v) = 2v - 1$ and $\beta(v) = 2v$. Also, $\alpha(y_i^*) = \beta(x_i^*)$ implies $y_i^* = x_i^* + \frac{1}{2}$ if $x_i^* \leq \frac{1}{2}$, and let $y_i^* = 1$ if $x_i^* > \frac{1}{2}$. Hence, for any $v_i^* \in [0,1]$

$$\eta(v_i|v_i^*) = \begin{cases} 2v_i, & v_i < v_i^* \\ 2v_i - 1, & v_i > v_i^* \end{cases}$$

and $\forall x_i^* \in [0,1]$

$$\delta(v_i|x_i^*) = \begin{cases} 2v_i, & \forall v_i < x_i^* \\ 2x_i^*, & \forall x_i^* \leq v_i \leq y_i^* \\ 2v_i - 1, & \forall v_i > y_i^* \end{cases}$$

To solve the profit-maximizing mechanism, we need to determine the vector x^* according to the algorithm constructed in the proof of Theorem 1. Without loss of generality, let $k_1 \leq k_2 \leq k_3$, then we must have $x_1^* \leq x_2^* \leq x_3^*$. We first consider the allocation rule $\tilde{q}$ which always breaks ties in favor of the participant with the highest index. We have

$$\begin{cases} \widetilde{Q}_1(x_1^*) &= (x_1^*)^2 - k_1 \\ \widetilde{Q}_2(x_2^*) &= x_2^*(x_2^* + \frac{1}{2}) - k_2 \\ \widetilde{Q}_3(x_3^*) &= (x_3^* + \frac{1}{2})^2 - k_3 \end{cases}$$

So we are looking for the (unique) vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$ which solves

$$\begin{cases} \widetilde{Q}_1(\tilde{x}_1) = \tilde{x}_1^2 - k_1 = 0 \\ \widetilde{Q}_1(\min(\tilde{x}_1, \tilde{x}_2)) + \widetilde{Q}_2(\tilde{x}_2) = \min(0, \tilde{x}_2^2 - k_1) + \tilde{x}_2(\tilde{x}_2 + \frac{1}{2}) - k_2 = 0 \\ \widetilde{Q}_1(\min(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)) + \widetilde{Q}_2(\min(\tilde{x}_2, \tilde{x}_3)) + \widetilde{Q}_3(\tilde{x}_3) = \\ \min(0, \tilde{x}_2^2 - k_1, \tilde{x}_3^2 - k_1) + \min(0, \tilde{x}_3(\tilde{x}_3 + \frac{1}{2}) - k_2) + (\tilde{x}_3 + \frac{1}{2})^2 - k_3 = 0 \end{cases}$$

Next, we let $x_1^* = \min(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, $x_2^* = \min(\tilde{x}_2, \tilde{x}_3)$ and $x_3^* = \tilde{x}_3$,

Depending on the respective values on k_1, k_2 and k_3, there are four different possibilities: (i) $x_1^* < x_2^* < x_3^*$, (ii) $x_1^* = x_2^* < x_3^*$, (iii) $x_1^* < x_2^* = x_3^*$, and (iv) $x_1^* = x_2^* = x_3^*$. Figure 1 illustrates the different set of values for k_1, k_2 and k_3 for which these different cases occur.

In area (i), which is determined by $k_2 > k_1 + \frac{1}{2}\sqrt{k_1}$ and $k_3 > \left(\frac{1}{4} + \sqrt{\frac{1}{16} + k_2}\right)^2$, we have

$$x_1^* = \sqrt{k_1} < x_2^* = \sqrt{\frac{1}{16} + k_2} - \frac{1}{4} < x_3^* = \sqrt{k_3} - \frac{1}{2}$$

In area (ii), which is determined by $k_1 \le k_2 \le k_1 + \frac{1}{2}\sqrt{k_1}$ and $k_3 > \left(\frac{3}{8} + \sqrt{\frac{1}{64} + \frac{k_1+k_2}{2}}\right)^2$, we have

$$x_1^* = x_2^* = \sqrt{\frac{1}{64} + \frac{k_1 + k_2}{2}} - \frac{1}{8} < x_3^* = \sqrt{k_3} - \frac{1}{2}$$

In area (iii), which is determined by $k_1 \le \frac{3-\sqrt{5}}{8}$ and $k_1 + \frac{1}{2}\sqrt{k_1} < k_2 \le k_3 \le \left(\frac{1}{4} + \sqrt{\frac{1}{16} + k_2}\right)^2$, we have

$$x_1^* = \sqrt{k_1} < x_2^* = x_3^* = \sqrt{\frac{1}{64} + \frac{k_2 + k_3}{2}} - \frac{3}{8} < \frac{1}{2}$$

In area (iv), which corresponds to all cases other than (i), (ii) and (iii), we have

$$x_1^* = x_2^* = x_3^* = \frac{\sqrt{5} - 1}{4} \quad (\frac{1}{4} < x_i^* < \frac{1}{2})$$

In cases (ii), (iii) and (iv), there will be a positive probability of ties. In case (iii), for instance, there is a $\frac{1}{4}$ probability that both 2 and 3 declare valuations between x_2^* and $y_2^* = x_2^* + \frac{1}{2}$. In case of a tie between 2 and 3, let p denote the probability that the tie is broken in favor of 2. If $p = 0$, then $Q_2(x_2^*) = \widetilde{Q}_2(x_2^*) \le 0 \le \widetilde{Q}_3(x_3^*) = Q_3(x_3^*)$; if $p = 1$, we have $Q_3(x_1^*) = \widetilde{Q}_2(x_2^*) + k_2 - k_3 \le 0 \le \widetilde{Q}_3(x_3^*) + k_3 - k_2 = Q_2(x_2^*)$. So there exists a value for p such that $Q_2(x_2^*) = Q_3^*(x_3^*) = 0$. In particular, let $k_1 = 0$, $k_2 = \frac{1}{3}$ and $k_3 = \frac{2}{3}$. In this case, we have $x_1^* = 0$ and $x_2^* = x_3^* = \frac{\sqrt{33}-1}{8}$. (Note that the worst-off types of participants 2 and 3 are the same although $k_3 = 2k_2$.) It is easy to find that $p = \frac{7-\sqrt{33}}{12}$. We see that the randomization rule is used here as an instrument in the design of an optimal allocation. Following a similar argument, we can find a randomization in case of a tie in cases (ii) and (iv).

Appendix

Proof of Lemma 1. If the trading mechanism $\{q,t\}$ is incentive compatible, i.e., for any two valuations $v_i, v_i^* \in [\underline{v}, \overline{v}]$,

$$U_i(v_i) = v_iQ_i(v_i) - T_i(v_i) \geq v_iQ_i(v_i^*) - T_i(v_i^*)$$

and

$$U_i(v_i^*) = v_i^*Q_i(v_i^*) - T_i(v_i^*) \geq v_i^*Q_i(v_i) - T_i(v_i)$$

These two inequalities imply that

$$(v_i - v_i^*)Q_i(v_i) \geq U_i(v_i) - U_i(v_i^*) \geq (v_i - v_i^*)Q_i(v_i^*)$$

Thus, if $v_i > v_i^*$, we must have $Q_i(v_i) \geq Q_i(v_i^*)$, so $Q_i(v_i)$ is non-decreasing. Furthermore, the above inequalities also imply that $U_i(v_i)$ is absolutely continuous, thus differentiable almost everywhere with derivative $\frac{dU_i}{dv_i}(v_i) = Q_i(v_i)$; or in the more convenient integral form

$$U_i(v_i) = U_i(v_i^*) + \int_{v_i^*}^{v_i} Q_i(u)\,du \tag{16}$$

Also, if $v_i^* \in V_i^*(q_i)$, by definition of $V_i^*(q_i)$, we have $U_i(v_i) - U_i(v_i^*) = \int_{v_i^*}^{v_i} Q_i(u)\,du \geq 0$; that is, the expected net utility $U_i(v_i)$ is minimized at v_i^*, hence $\{q,t\}$ is individually rational if and only if $U_i(v_i^*) \geq 0$.

Suppose now that $Q_i(v_i)$ is non-decreasing and $U_i(v_i)$ satisfies (16) for some $v_i^* \in V_i^*(q_i)$, then for any v_i and $\hat{v}_i$,

$$\begin{aligned} U_i(v_i) - U_i(\hat{v}_i) &= \int_{\hat{v}_i}^{v_i} Q_i(u)\,du \\ &\geq (v_i - \hat{v}_i)Q_i(\hat{v}_i) \end{aligned}$$

where the inequality follows from the fact that $Q_i(u)$ is non-decreasing in u. This inequality can be rewritten as

$$U_i(v_i) \geq U_i(\hat{v}_i) + (v_i - \hat{v}_i)Q_i(\hat{v}_i) = v_iQ_i(\hat{v}_i) - T_i(\hat{v}_i)$$

Thus, $\{q,t\}$ is incentive compatible. In the above, we have already shown that when $U_i(v_i^*) \geq 0$, an incentive compatible mechanism is also individually rational. □

Proof of Lemma 2. From Lemma 1, if $\{q,t\}$ is a feasible trading mechanism, the expected revenue of the mechanism equals

$$R = E\left[\sum_{i=1}^{n} t_i(v)\right]$$

$$
\begin{aligned}
&= E\left[\sum_{i=1}^{n}(v_i Q_i(v_i) - U_i(v_i))\right] \\
&= E\left[\sum_{i=1}^{n}(v_i Q_i(v_i) - \int_{v_i^*}^{v_i} Q_i(u)\,du)\right] - \sum_{i=1}^{n} U_i(v_i^*)
\end{aligned}
$$

Integrating the second term on the right by parts, we obtain

$$R = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)Q_i(v_i)] - \sum_{i=1}^{n} U_i(v_i^*)$$

For any function $q(v)$ such that $q_i(v_i, v_{-i})$ is non-decreasing in v_i, from the above and individual rationality constraints, the maximum expected revenue from any feasible mechanisms implementing $q(v)$ cannot be greater than

$$R(q) = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)Q_i(v_i)]$$

To complete the proof, we must construct a payment function $t(v)$ so that $\{q, t\}$ is a feasible trading mechanism leading $R(q)$. There are many such functions which could be used; we will consider a function defined as follows:

$$t_i(v) = E_{-i}\left[v_i q_i(v_i, v_{-i}) - \int_{v_i^*}^{v_i} q_i(u, v_{-i})\,du\right] - k_i v_i^*$$

Note that the payment by trader i depends only on his own valuation v_i. From Lemma 1, the mechanism $\{q, t\}$ is incentive compatible and individually rational, and the expected revenue from this mechanism equals $R(q)$. Thus, our proof of Lemma 2 is complete. □

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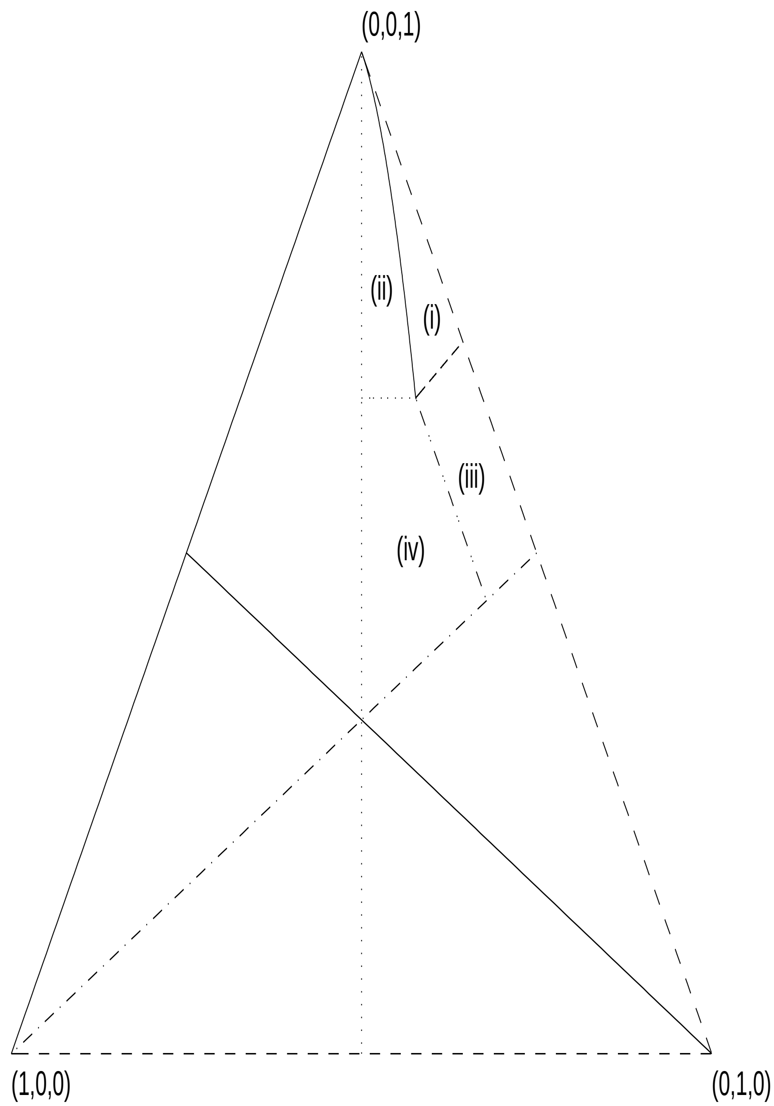

Figure 1−−Four different bunching regions

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