97s-12

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Montréal
Mars 1997

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AggregationEquilibrium Asset Prices and No-Arbitrage with Portfolio Constraints[*]

Jérôme B. Detemple[†], Shashidhar Murthy[‡]

Résumé / Abstract

Cet article examine l'évaluation intertemporelle des titres financiers lorsque les ventes à découvert sont limitées en proportion à la valeur du portefeuille de l'investisseur. Le prix de tout actif dépasse, pour tout investisseur, la valorisation de ses dividendes basée sur l'utilité marginale de consommation individuelle, lorsque chaque investisseur se trouve contraint dans un actif quelconque dans un état quelconque; nous démontrons l'existence d'un tel équilibre. Le prix d'un actif a trois composantes : la valeur de consommation de ses dividendes, une prime de valeur spéculative, et une prime de valeur de collatéral. La validité de l'approche de validation fondée sur l'absence d'arbitrage dépend de manière critique de la différence entre un actif réel et sa contrepartie synthétique.

We examine intertemporal asset pricing when short sales are constrained in proportion to the value of an investor's portfolio. All assets' prices exceed every investor's marginal utility of consumption-based valuation of the associated dividends if every investor finds himself constrained in some asset in some state; we exhibit such an equilibrium. An asset's price decomposes into three (investor-specific) components: the consumption-value of its dividends, a speculative-value premium, and a collateral-value premium. The validity of the no-arbitrage pricing approach is shown to depend critically on the difference between real securities and their synthetic counterparts.

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This paper examines the equilibrium asset pricing implications of frictions which take the form of constraints on portfolio weights that limit short sales of assets. Such constraints differ from exogenous bounds on short sales in that they capture an investor's ability to increase his borrowing or short sales as a function of his "credit-worthiness." Margin requirements faced by most investors are one example of such portfolio constraints. Another example is provided by limitations on leverage which often apply to portfolio managers.

In standard models of frictionless economies, the price of a traded claim to a future payoff is equal to the expectation of the product of the payoff and a discount factor. Furthermore, this discount factor can be identified with the relevant ratio of the marginal utilities of consumption of any investor - his intertemporal marginal rate of substitution (IMRS). It is increasingly recognized that understanding the implications of frictions is important; in their presence, asset prices may deviate from IMRS-based valuations of dividends. For instance, Luttmer (1996) and He and Modest (1995) characterize bounds on such deviations based on an IMRS computed with aggregate consumption and the preferences of an aggregate agent, following Hansen and Jagannathan (1991).

In this paper, we investigate a finite horizon general equilibrium model with portfolio constraints. We show that equilibrium prices of assets exceed every investor's IMRS-based valuation of the associated payoffs by "speculative-value" and/or "collateral-value" premia in the presence of binding constraints on portfolio weights. We also study the pricing of derivative securities in this context, motivated by the suggestion that the preponderance of regulatory constraints is an important reason for the success of many financial innovations (Miller (1986)). The validity of standard methods for pricing derivatives is shown to depend critically on the difference between "real" securities and their "synthetic" counterparts.

The existence of a speculative premium on an asset subject to a no-short-sales constraint was shown by Harrison and Kreps (1978) (hereafter, HK). In an infinite horizon partial equilibrium setting, they demonstrate an equilibrium in which the stock price exceeds all agents' valuations of the associated dividends. HK characterize investor behavior in their model as speculative: an investor willingly pays more than his valuation of the dividends because he expects to be able to resell the asset at a favorable price. A similar result is also reported by Allen, Morris and Postlewaite (1993) (hereafter, AMP) in a finite horizon general equilibrium model with a binding no-short-sales constraint. It is straightforward to show that these results extend to the case of limited short sales when exogenous bounds are placed on the numbers of shares that may be short sold, and to the case of multiple assets. In particular, when such short sales constraints are binding on every investor for some assets, at the current date or in some future state, only these assets may contain speculative premia. All other assets' prices will in equilibrium equal every investor's intertemporal marginal utility of consumption-based valuation of the associated dividends.

In contrast to exogenous bounds on the number of shares sold short, the negative lower bound constraints on portfolio weights that we consider allow short sales in proportion to the value of an investor's

portfolio. In an intertemporal model of investors with heterogeneous beliefs, one stock and a riskless asset (in zero net supply), we exhibit an equilibrium in which short sales and leverage constraints bind on all investors. Since all investors have logarithmic utilities with a common rate of time preference, the binding constraints impact equilibrium through their effects on the riskfree interest rate and the market price of risk, while the stock price is invariant to the constraints. However, investors value the stock (and every tradable claim) for more than the consumption its dividends provide. We prove that the equilibrium price S of the stock with dividends D has the decomposition

$$S_t = E_t^i[\int_t^T m_{tu}^i(D_u + \chi_u^i + \kappa_u^i)du] \tag{1}$$

where m_{tu}^i is investor i's IMRS between dates t and u, E_t^i is his expectation operator conditional on information at time t, and $\chi^i \geq 0$, $\kappa^i \geq 0$. The first component is the familiar IMRS-based valuation of the dividends D by investor i, which we term the stock's "consumption-value." The process $\chi^i > 0$ in a given event (ω,t) if and only if investor i is short sales-constrained in the stock in the same event whereas $\kappa^i > 0$ in (ω,t) if and only if either the constraint on short sales of stock or the leverage constraint binds on investor i in (ω,t). We show that, at the margin, the purchase of a share of the stock enables investor i to realize expected "speculative gains" at the rate χ^i, in excess of foregone capital gains and dividends, by selling it to acquire the riskless asset in a timely manner. Thus, along the lines of HK (1978) and AMP (1993), we interpret $E_t^i[\int_t^T \chi_u^i m_{tu}^i du]$ as a speculative-value premium in S relative to investor i's IMRS: his willingness to pay more than the consumption value to him of the dividends is partially explained by the positive probability of a resale of the stock to another investor at a favorable price. We also show that, at the margin, ownership of a share of the stock enables investor i to earn expected speculative gains at the rate κ^i from the timely *short* sales of the stock *or* the riskless asset. In other words, ownership of the stock supports the valuable option of being able to borrow or short sell. Hence, we interpret κ^i as the "collateral services" provided by the stock for which investor i is willing to pay the ex ante premium $E_t^i[\int_t^T \kappa_u^i m_{tu}^i du]$. The presence of the latter, collateral-value premium complements the results of HK (1978) and AMP (1993). The sources of value from an investment in the riskless asset may also be categorized as in (1).

The magnitudes of the speculative gains and collateral services that investor i imputes to an asset depend on both how much short sales and leverage is admissible and the incidence of these constraints on him. A "local" analysis of the events (ω,t) in which the constraint on the stock binds, for instance, reveals that the speculative gains $\chi^i(\omega,t)$ from the stock increase in the stringency of the constraint. The collateral services $\kappa^i(\omega,t)$ from the stock, however, have an inverted U-shape with respect to the degree of the restriction. This reflects the fact that there is clearly no value to collateral when no short sales are permitted *and* also when unlimited short sales are allowed. Hence, agent i's expected gains from trade in the stock, which equal the

2

sum of the speculative gains and the collateral services $\chi^i(\omega,t) + \kappa^i(\omega,t)$, are not necessarily decreasing as the constraint is relaxed. Stated differently, this result also has the implication that observed deviations of returns from IMRS-based predictions can be non-monotone in the degree of stringency of a short sales constraint. This is interesting because it is at odds with the intuition that there must be smaller expected gains from trade the more unfettered are investors.

Our analysis uses results of Cvitanic and Karatzas (1992) on the individual consumption-portfolio optimization problem under constraints on portfolio choice. The general equilibrium framework with log utilities and heterogeneous beliefs that we employ extends Detemple and Murthy (1994) to a single dividend-paying stock and a class of constraints. It should be stressed at the very outset that the assumption of differing priors is not the key to the generation of speculative and/or collateral-value premia; rather, it is the presence of portfolio constraints that bind in equilibrium. Of course, the latter requires the presence of enough heterogeneity in order to induce trade: the differing priors serve this purpose. We also extend the decomposition result in (1) to economies with (i) a finite number $I > 2$ of agents, (ii) heterogeneous von Neumann Morgenstern utilities, and (iii) multiple assets in positive net supply, taking as given the existence of an equilibrium in which *every* investor finds himself constrained in *some* asset at either the current date or some future date with positive probability. Hence, all assets sell at prices above all investors' valuations of the associated dividends provided one asset does; this is in contrast to the no-short-sales case of HK (1978) and AMP (1993). However, the magnitudes of the consumption, speculative and collateral value components for a given asset differ across investors.

When securities have different degrees of collateralizability and a solvency constraint binds on a single investor who supports prices, Hindy (1995) proved that an asset's price admits a decomposition into a dividend-based value and a residual that depends on the asset's collateralizability. Any difference in the prices of two trading strategies that have the same cashflows may be interpreted as a value imputed to their differing collateral services. We do not allow for the existence of multiple assets with the same cashflows but different prices; nevertheless, our equilibrium analysis in terms of multiple investors' marginal utilities enables the identification of *investor-specific* collateral-value *and* speculative-value components in the price of an asset. This suggests that in any equilibrium in which assets are valued for speculative and/or collateral reasons, a "market" value associated with these asset characteristics will not coincide with individual investors' valuations of the same. Furthermore, our results show that an asset cannot have value as collateral for any investor unless he also values some other asset for speculative reasons.

We also examine the implications of our general equilibrium model for the valuation of derivatives. A recent literature addresses the implications of pricing by no-arbitrage in settings where constraints on short sales or leverage limit the class of attainable claims, i.e. payoffs which may be synthesized using admissible

portfolio strategies. Typically, these papers show that no-arbitrage yields bounds on the prices of unattainable claims; the bounds collapse to yield a single price if and only if the claim is attainable. We show that interpretations of these results are sensitive to the difference between "synthetic" claims and their "real" counterparts. Under binding portfolio constraints, the introduction of markets in some real securities - claims whose payoffs may be otherwise synthesized - has an impact on equilibrium. These real securities have payoffs whose synthesis involves an asset in which some investor is constrained, equivalently, an asset with a speculative-value premium. Markets in such derivatives allow at least one investor to circumvent, completely or partially, any previously binding portfolio constraints, and this has an equilibrium impact. Hence, it may not be appropriate to price derivatives by no-arbitrage even if the primitive assets' prices and prevailing constraints on trading strategies suggest that the derivatives can be replicated by admissible portfolios. In other words, the set of redundant securities may be a (proper) subset of the class of attainable claims, in contrast to the case of frictionless markets where every attainable claim is redundant.[1,2]

The paper is organized as follows. Section 1 presents the general equilibrium model with one stock. In section 2 we solve for competitive equilibrium in the presence of binding constraints. Section 3 establishes that the excess of an asset's price over its consumption-value is comprised of speculative-value and collateral-value components, and examines their properties. The implications for pricing by no-arbitrage are developed in section 4. Extensions of the results are presented in section 5. Concluding remarks are formulated in section 6. All proofs are contained in the appendix.

1. A General Equilibrium Model

We consider a continuous time economy with a finite horizon $[0,T]$. There is 1 share outstanding of a dividend-paying stock and a single consumption good which serves as numeraire. The drift of the dividend growth rate is stochastic and unobservable. Investors form beliefs about the changing growth rate based on the information contained in realized dividends and on a second information source which can be interpreted as publicly released analysts' forecasts.

Formally, the uncertainty is represented by a complete probability space $(\Omega, \Im, P)$ on which is defined a $\mathbf{R}^3$-valued Brownian Motion process $W = (W_0, W_1, W_2)$. Let $\Im_{(\cdot)}^W$ denote the augmented filtration generated by W. Also define the filtration $\mathfrak{C} \times \Im_{(\cdot)}^W$ and its augmentation $\Im_{(\cdot)}$, where $\mathfrak{C}$ is a sigma-algebra independent of $\Im_T^W$,

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[1]The relationship between this result and those of Heath and Jarrow (1987), Grossman (1988), Back (1993), and Chen (1995) is discussed in section 4.

[2]Other studies on the role of frictions in asset pricing include Black (1973), Ross (1977), Jarrow (1980), Dybvig and Ross (1986), Prisman (1986), Scheinkman and Weiss (1986), He and Pages (1993), Jouini and Kallal (1995a, 1995b), and Cuoco (1997).

and set $\Im = \Im_T$. There are two types of agents in the economy, indexed by i, i = 1,2, who are each endowed with shares of the stock. The stock pays dividends at the (observable) rate D > 0 per unit time where D solves the stochastic differential equation,

$$D_t = D_0 + \int_0^t D_s \mu_s ds + \int_0^t D_s \sigma_s(D^s) dW_{1s}, \quad t \in [0,T]; \quad D_0 > 0 \text{ given}, \tag{2}$$

with

$$\mu_t = \mu_0 + \int_0^t H_s ds + \int_0^t b_s(D^s) dW_{0s}, \quad t \in [0,T]. \tag{3}$$

Here σ and μ represent, respectively, the volatility and drift of the dividend growth rate; $D^s \equiv (D_t; t \in [0,s])$ is the trajectory of D up to time s. The initial value μ_0 is a $\mathfrak{C}$-measurable random variable that is independent of dividend realizations, and more generally, of W. The processes b and H represent, respectively, the volatility and drift coefficients of the process μ. Agents observe the realized dividends and know the functionals $\sigma(.)$ and $b(.)$. The processes μ, H and W are unobserved.

Agents also observe the realizations Y (analysts' forecasts) which are generated by the information technology,

$$Y_t = \int_0^t A_u du + \int_0^t s_u(Y^u) dW_{2u}, \quad t \in [0,T]. \tag{4}$$

The components A and s represent the drift and volatility, respectively, of the information process; $Y^u \equiv (Y_t; t \in [0,u])$ is the trajectory of Y up to time u. Agents know the functional $s(\cdot)$ and do not observe the process A. The observations Y convey information about the drift of the dividend growth rate μ to the extent that the covariation of A and μ differs from zero. The degree of informativeness of the forecasts Y is jointly determined by the covariation of the processes A and μ, and the (inverse) precision s. The common information of agents is summarized by the filtration $\Im_{()}^{D,Y}$ generated by the pair (D,Y). Since for each t $\in$ [0,T] we have $\Im_t^{D,Y} \subset \Im_t$ agents have incomplete information; thus, processes such as μ and W, which are adapted to $\Im_{()}$ but not to $\Im_{()}^{D,Y}$ are unobserved.[3]

Agents have heterogeneous beliefs represented by probability measures P^i, i = 1,2, that are equivalent to P. The measures P^1, P^2 and P coincide everywhere except on $\mathfrak{C}$. Let $E^i(\cdot)$ denote the expectation operator relative to P^i, i = 1,2. Investor i's posterior beliefs about future dividends may be described in terms of the $\Im_{()}^{D,Y}$-measurable conditional expectations $\hat{\mu}_t^i = E^i[\mu_t | \Im_t^{D,Y}]$, $\hat{A}_t^i = E^i[A_t | \Im_t^{D,Y}]$ and $\hat{H}_t^i = E^i[H_t | \Im_t^{D,Y}]$, which depend on the history (D^t,Y^t) and on prior beliefs (Liptser and Shiryayev (1977), Lemma 4.9). Agent i's estimate of the rate of dividend growth, $\hat{\mu}_t^i = E^i[\mu_t | \Im_t^{D,Y}]$, satisfies

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[3]We assume that the $\mathbf{R}_+$-valued functionals σ, b and s are bounded above and away from zero for all t $\in$ [0,T]. See Detemple and Murthy (1994) for the appropriate technical conditions on H, A, σ, b, s, and μ_0, that ensure the existence of a unique strong solution to (2)-(4). Also, $\mu_0 \in \mathfrak{L}^2$ implies that $(\mu_t, D_t) \in \mathfrak{L}^2 \times \mathfrak{L}^2$ for all t $\in$ [0,T].

$$\hat{\mu}^i = \hat{\mu}_0^i + \int_0^t [\hat{H}_s^i ds + n_s^i \sigma_s^{-1} dv_{1s}^i + v_s^i s_s^{-1} dv_{2s}^i], \quad i=1,2, \tag{5}$$

where $n_t^i \equiv E^i[\mu_t^2 \mid \mathfrak{I}_t^{D,Y}] - (\hat{\mu}_t^i)^2$ and $v_t^i \equiv E^i[\mu_t A_t \mid \mathfrak{I}_t^{D,Y}] - \hat{\mu}_t^i \hat{A}_t^i$ represent the estimation errors associated with dividend growth information and with the information technology (4), respectively. The processes v_1^i and v_2^i are defined as,

$$v_{1t}^i \equiv \int_0^t \sigma_v^{-1}[dD_v/D_v - \hat{\mu}_v^i dv] \quad \text{and} \quad v_{2t}^i \equiv \int_0^t s_v^{-1}[dY_v - \hat{A}_v^i dv], \tag{6}$$

for $i = 1,2$. We suppose that $\hat{\mu}^i$ and $\hat{A}^i$ satisfy appropriate Lipschitz and growth conditions. This ensures that v_1^i and v_2^i are innovation processes induced by the individual beliefs of agent i, i = 1,2, and the observation processes (2) and (4) respectively; i.e., the information filtration $\mathfrak{I}_{()}^{D,Y}$ coincides with the filtration $\mathfrak{I}_{()}^i$ generated by $v^i = (v_1^i, v_2^i)$ (Liptser and Shiryayev (1977), Lemma 4.9). We also assume that n^i and v^i are bounded away from zero. The well known linear-Gaussian case is an example of an uncertainty-information structure which satisfies (2)-(6). Let $E_t^i(\cdot)$ denote the expectation operator relative to P^i conditional on information $\mathfrak{I}_t^i$, $t \in [0,T]$, i = 1,2.

There are competitive markets in shares of the stock, the consumption good and in an instantaneous riskless bond which is in zero net supply. Agents take as given an (ex-dividend) stock price process S and an instantaneous interest rate process r, which must be $\mathfrak{I}_{()}^i$-progressively measurable since all agents have (common) information $\mathfrak{I}_{()}^i$. Hence, the cum-dividend stock returns and the cumulative riskless rate of return R have the respective representations,

$$dS_t + D_t dt = S_t \alpha_t^i dt + S_t \rho_{1t} dv_{1t}^i + S_t \rho_{2t} dv_{2t}^i, \quad t \in [0,T]; \quad S_T = 0; \quad i = 1,2, \tag{7}$$

$$dR_t = r_t dt, \quad t \in [0,T]; \quad R_0 = 0, \tag{8}$$

where α^i, ρ_1, ρ_2 and r are $\mathfrak{I}_{()}^i$-progressively measurable, integrable and are determined in equilibrium along with the initial stock price S_0. Furthermore, the consistency condition $\alpha_t^2 = \alpha_t^1 + \rho_{1t}\sigma_t^{-1}(\hat{\mu}_t^2 - \hat{\mu}_t^1) + \rho_{2t}s_t^{-1}(\hat{A}_t^2 - \hat{A}_t^1)$, $t \in [0,T]$, which is implied by (6)-(7), must hold since both agents face the same prices.

Consider the space of nonnegative processes that are progressively measurable with respect to $\mathfrak{I}_{()}^{D,Y}$ and for which the expectation in (9) is finite. Agents' preferences are defined on this space and have the von Neumann-Morgenstern representation,

$$U^i(c^i) = E^i[\int_0^T e^{-\beta t} \text{Log}(c_t^i) dt], \tag{9}$$

where β is a constant subjective discount rate which is identical across agents. Agent i is endowed with $x^i > 0$ shares of the stock, i = 1,2; the total supply of shares $x^1 + x^2$ equals 1.

Let π_t^i represent the proportion of agent i's wealth X_t^i that is invested in the stock at time t; $X_t^i(1-\pi_t^i)$ is invested in the riskless asset. We assume agent i = 1,2 is subject to the following constraints on his portfolio

6

weights which (i) limit admissible short sales of the stock and (ii) restrict borrowing at the riskless rate:

$$\pi_t^i \geq -\xi_S; \quad \xi_S \in [0,\infty),$$
$$1-\pi_t^i \geq -\xi_B; \quad \xi_B \in [0,\infty),$$

(10)

where ξ_S and ξ_B are two nonnegative constants. Equivalently, these 2 constraints can be summarized by the single, 2-sided constraint: $\pi_t^i \in [-\xi_S, 1+\xi_B]$, $t \in [0,T]$, $i = 1,2$. An interpretation in terms of margin requirements is as follows: an investor may borrow to buy the stock provided his borrowing does not exceed $\xi_B/(1+\xi_B)$ of the value of the stock. Similarly, going short the stock and lending the proceeds of the short sale at the interest rate requires an out-of-pocket investment in the riskless asset of at least $1/\xi_S$ of the stock's value.

A pair of consumption and portfolio processes $(c^i,\pi^i) \in \mathcal{L}_+[0,T] \times \mathcal{L}^2[0,T]$ for agent i is admissible at the prices (7)-(8) if it is $\mathfrak{I}_{(\cdot)}$-progressively measurable, and if it satisfies the portfolio constraints (10) and the nonnegativity restriction $X_t^i \geq 0$, $t \in [0,T]$, where X^i solves the stochastic differential equation,

$$dX_t^i = (r_t X_t^i - c_t^i)dt + X_t^i\pi_t^i(\alpha_t^i-r_t)dt + X_t^i\pi_t^i[\rho_{1t}dv_{1t}^i+\rho_{2t}dv_{2t}^i], \quad t \in [0,T],$$

(11)

with $X_0^i = x^i S_0$. A pair of admissible processes (c^i,π^i) is optimal for the preferences U^i if there is no other admissible pair $(c^{i'},\pi^{i'})$ such that $U^i(c^{i'}) > U^i(c^i)$. A rational expectations equilibrium is a collection $((c^i,\pi^i),$ $i = 1,2)$ of admissible consumption-portfolio strategies, a stock price process S and an interest rate process r such that (c^i,π^i) is optimal for agent i and all markets clear: $\Sigma_i c^i = D$, $\Sigma_i \pi^i X^i = S$, and $\Sigma_i(1-\pi^i)X^i = 0$.

2. Equilibrium with Binding Constraints

2.1 Optimal Consumption-Portfolio Policies

The consumption-portfolio choice problem of an individual agent in the economy of section 1 is equivalent to an optimization problem subject to two types of constraints: (i) portfolio constraints on existing assets, and (ii) an incomplete markets constraint. We solve this decision problem by embedding it in a larger family of "unconstrained" problems, following the approach of Cvitanic and Karatzas (1992), as detailed in the Appendix. This yields the following optimal demand functions for the economy of section 1:

Theorem 2.1: Define $\rho_t \equiv ((\rho_{1t})^2+(\rho_{2t})^2)^{\frac{1}{2}} > 0$, $mpc_t \equiv [\int_t^T e^{-\beta(s-t)}ds]^{-1}$, and suppose that for $t \in [0,T]$, $i = 1,2$,

$$(\alpha^i,r,\rho_1,\rho_2) \in (\mathcal{L}^2[0,T])^4 \quad \text{and} \quad E^i\{\exp(\frac{1}{2}\max[(1+\xi_B)^2,(\xi_S)^2]\int_0^t(\rho_u)^2du)\} < \infty.$$

(12)

In the economy of section 1, agent i's wealth solves (11) with the optimal demands, for $i = 1,2$:

$$c_t^i = mpc_t X_t^i, \quad t \in [0,T]$$

(13)

$$\pi_t^i = \max\{-\xi_S,\min\{(\rho_t)^{-2}[\alpha_t^i-r_t],1+\xi_B\}\}, \quad t \in [0,T]$$

(14)

The assumption of logarithmic utility simplifies the above solution which is characterized by a state-independent marginal propensity to consume out of wealth and a myopic optimal portfolio rule. When neither the borrowing nor the short sales constraint is binding on agent i, his optimal portfolio is given by the standard mean-variance demand: $\pi_t^i = (\rho_t)^{-2}[\alpha_t^i - r_t]$. In events in which $(\rho_t)^{-2}[\alpha_t^i - r_t] < -\xi_S$, the short sales constraint binds and agent i's optimal demand for the stock is given by $\pi_t^i = -\xi_S$. Similarly, when $(\rho_t)^{-2}[\alpha_t^i - r_t] > 1+\xi_B$, the borrowing constraint binds: $1-\pi_t^i = -\xi_B$. Combining these three cases yields the constrained optimal portfolio demand in (14). Observe that the latter may also be written as $\pi_t^i \equiv (\rho_t)^{-2}(\alpha_t^i + \lambda_{1t}^i - r_t)$ where

$$\lambda_{1t}^i = \begin{cases} 0 & \text{if } (\rho_t)^{-2}(\alpha_t^i - r_t) \in [-\xi_S, 1+\xi_B] \\ -(\rho_t)^2 \xi_S - \alpha_t^i + r_t & \text{if } (\rho_t)^{-2}(\alpha_t^i - r_t) < -\xi_S, \\ (\rho_t)^2 (1+\xi_B) - \alpha_t^i + r_t & \text{if } (\rho_t)^{-2}(\alpha_t^i - r_t) > 1+\xi_B. \end{cases} \quad (15)$$

The Kuhn-Tucker multipliers $\max(\lambda_{1t}^i, 0)$ and $\max(-\lambda_{1t}^i, 0)$ which are associated with the constraint on short sales of the stock and bond, respectively, represent the corresponding "shadow prices" for investor i.

2.2 Equilibrium

We now provide explicit solutions for equilibrium prices and allocations after stating two technical assumptions. As will be clear, the equilibrium interest rate in this setting exhibits discontinuities. Assumption 2.1 below restricts the interest rate to the class of processes which are left continuous with right limits (LCRL). We need Assumption 2.2 to have a well defined equivalent martingale measure in equilibrium.

Assumption 2.1: The equilibrium interest rate process r is LCRL.

Assumption 2.2: $E^i[\exp(\frac{1}{2}\int_0^T (\sigma_s^{-1} | \hat{\mu}_s^1 - \hat{\mu}_s^2 | + \sigma_s)^2 ds)] < \infty$, i = 1,2.

Define $\Delta \equiv (\hat{\mu}^1 - \hat{\mu}^2)/\sigma^2$, the difference in expected dividend growth rates across agents per unit squared volatility. Since $\hat{\mu}$ in (5)-(6) and σ are Ito processes, Δ has a stochastic differential representation of the form,

$$d\Delta \equiv \mu^{\Delta i} dt + \sigma_1^{\Delta} dv_{1t}^i + \sigma_2^{\Delta} dv_{2t}^i; \quad i = 1,2, \quad (16)$$

for some $\mu^{\Delta i}$, σ_1^{Δ} and σ_2^{Δ}. Part (i) of the following assumption will be sufficient for the constraints to bind in equilibrium. The restriction in part (ii) on initial endowments x^i, i = 1,2, and the constants ξ_S and ξ_B will ensure, without loss of generality, that the constraints do not bind at time 0.

<u>Assumption 2.3</u>:

(i) Consider the representation of Δ in (16). We suppose there exists an open set in $\mathbf{R}$ containing $[-1-\xi_S-\xi_B, 1+\xi_S+\xi_B]$ such that for all Δ in this set and for all $t \in [0,T]$: $(\mu_t^{\Delta i}, \sigma_{1t}^{\Delta}, \sigma_{2t}^{\Delta}) \neq (0, 0, 0)$ P^i-a.s.

(ii) Also assume the initial condition: $-\min[\xi_B/x^1,(1+\xi_S)/x^2] \leq \Delta_0 \leq \min[\xi_B/x^2,(1+\xi_S)/x^1]$.

In order to describe the competitive equilibrium it is useful to delineate events according to the incidence of the constraints on the agents. Since we have 2 agents and 2 constraints there are 7 possible configurations (including boundary cases of relevance) which are defined using the sets,

$$L_0 \equiv \{(a,b) \in \mathbf{R}x[0,1]: -\min[\xi_B b^{-1},(1+\xi_S)(1-b)^{-1}] \leq a \leq \min[\xi_B(1-b)^{-1},(1+\xi_S)b^{-1}]\}$$

$$L_{1B} \equiv \{(a,b) \in \mathbf{R}x[0,1]: \xi_B(1-b)^{-1} < a, \text{ and } b < (1+\xi_S)/(1+\xi_S+\xi_B)\}$$

$$L_{1S} \equiv \{(a,b) \in \mathbf{R}x[0,1]: a < -(1+\xi_S)(1-b)^{-1}, \text{ and } b < \xi_B/(1+\xi_S+\xi_B)\}$$

$$L_{2B} \equiv \{(a,b) \in \mathbf{R}x[0,1]: a < -\xi_B b^{-1}, \text{ and } \xi_B/(1+\xi_S+\xi_B) < b\} \tag{17}$$

$$L_{2S} \equiv \{(a,b) \in \mathbf{R}x[0,1]: (1+\xi_S)b^{-1} < a, \text{ and } (1+\xi_S)/(1+\xi_S+\xi_B) < b\}$$

$$L_{1B.2S} \equiv \{(a,b) \in \mathbf{R}x[0,1]: (1+\xi_S+\xi_B) < a, \text{ and } b = (1+\xi_S)/(1+\xi_S+\xi_B)\}$$

$$L_{1S.2B} \equiv \{(a,b) \in \mathbf{R}x[0,1]: a < -(1+\xi_S+\xi_B), \text{ and } b = \xi_B/(1+\xi_S+\xi_B)\}.$$

These sets are illustrated in Figure 1. For example, we will show that in events in which $(\Delta_t,\delta_t) \in L_{iB}$ (L_{iS}) agent i will be borrowing (short sales) constrained with agent $j \neq i$ unconstrained.

<div align="center">INSERT FIGURE 1 HERE</div>

The competitive equilibrium is described next:

<u>Theorem 2.2</u>: Consider the economy of section 1 and suppose that assumptions 2.1-2.3 hold. In equilibrium, the following events arise with positive probability,

    (i) $\{(\Delta_t,\delta_t) \in L_0$ for some $t \in [0,T]\}$: both agents are unconstrained,

    (ii) $\{(\Delta_t,\delta_t) \in L_{iB}$ for some $t \in (0,T]\}$: agent i is borrowing constrained with agent j unconstrained,

    (iii) $\{(\Delta_t,\delta_t) \in L_{iS}$ for some $t \in (0,T]\}$: agent i is short sales constrained with agent j unconstrained,

and   (iv) $\{(\Delta_t,\delta_t) \in L_{iB,jS}$ and $(\Delta_{t-},\delta_{t-}) \in L_{iB}$ (L_{jS}) for some $t \in (0,T]\}$: agent i is borrowing constrained and agent j is unconstrained (agent j is short sales constrained and agent i is unconstrained),

for i,j = 1,2, i $\neq$ j. The equilibrium price of the stock is,

$$S_t = (mpc_t)^{-1}D_t, \quad t \in [0,T]; \tag{18}$$

it satisfies (7) with $(\alpha^i, \rho_1, \rho_2) = (\hat{\mu}^i+\beta, \sigma, 0)$. The equilibrium interest rate is given by,

<div align="center">9</div>

$$r_t = \begin{cases} \beta + \delta_t\hat{\mu}_t^1 + (1-\delta_t)\hat{\mu}_t^2 - \sigma_t^2 & \text{for } (\Delta_t,\delta_t) \in L_0 \\ \beta + \hat{\mu}_t^i - \sigma_t^2 + \xi_B(X_t^j/X_t^i)\sigma_t^2 & \text{for } (\Delta_t,\delta_t) \in L_{iB} \\ \beta + \hat{\mu}_t^i - \sigma_t^2 - (1+\xi_S)(X_t^j/X_t^i)\sigma_t^2 & \text{for } (\Delta_t,\delta_t) \in L_{iS} \\ \lim_{s\uparrow t} r_s & \text{for } (\Delta_t,\delta_t) \in L_{iB,jS}, \end{cases} \tag{19}$$

for $i,j = 1,2$, $i \neq j$. The equilibrium state price density is $\eta_t^i \exp\{-\int_0^t r_s ds\}$ relative to P^i where

$$\eta_t^i \equiv \exp\{-\int_0^t \theta_{1s}^i dv_{1s}^i - (1/2)\int_0^t (\theta_{1s}^i)^2 ds\}, \tag{20}$$

is a $(P^i, \mathfrak{I}_{(.)}^i)$-martingale, and $\theta_{1t}^i \equiv \sigma_t^{-1}[\hat{\mu}_t^i + \beta - r_t]$ is the market price of risk. Equivalently, there exists an equivalent martingale measure (EMM), denoted Q, with density η^i relative to P^i, $i = 1,2$. The equilibrium consumption and portfolio allocations are:

$$c_t^i = mpc_t X_t^i \quad \text{and} \quad \pi_t^i = \max\{-\xi_S, \min[\sigma_t^{-2}(\hat{\mu}_t^i + \beta - r_t), 1 + \xi_B]\} \tag{21}$$

where the equilibrium distribution of wealth $\delta \equiv X^1/X$ has the representation $d\delta \equiv \mu^\delta dt + \sigma^\delta dv_1^i$ in (A.10) of the Appendix, and aggregate wealth is $X_t = S_0 \exp\{\int_0^t [\hat{\mu}_u^i - mpc_u - \frac{1}{2}(\sigma_u)^2] du + \int_0^t \sigma_u dv_{1u}^i\}$.

2.3 Some Properties of Equilibrium

In order to understand the structure of the equilibrium with the portfolio constraints it is useful to consider the benchmark case of the unconstrained, but otherwise identical, economy $(\xi_B \uparrow \infty, \xi_S \uparrow \infty)$.[4] The latter's competitive equilibrium is described below; the proof follows from that of Theorem 2.2.[5]

Corollary 2.1: In the "unconstrained" economy $(\xi_B \uparrow \infty, \xi_S \uparrow \infty)$, equilibrium prices and allocations for $t \in [0,T]$ are: $S_t = X_t = (mpc_t)^{-1} D_t$, $r_t^u = \delta_t^u \hat{r}_t^1 + (1-\delta_t^u)\hat{r}_t^2$, $c_t^{iu} = mpc_t X_t^{iu}$, $\pi_t^{iu} = \sigma_t^{-2}(\hat{\mu}_t^i + \beta - r_t^u)$ where $\hat{r}_t^i = \beta + \hat{\mu}_t^i - \sigma_t^2$, $i = 1,2$, and the distribution of wealth is $\delta_t^u \equiv X_t^{1u}/X_t = x^1 \exp[\int_0^t \sigma_s^{-1}(1-\delta_s^u)(\hat{\mu}_s^1 - \hat{\mu}_s^2) dv_{1s}^1 + (1/2)\int_0^t [\sigma_s^{-1}(1-\delta_s^u)(\hat{\mu}_s^1 - \hat{\mu}_s^2)]^2 ds]$.

A comparison of Theorem 2.1 and Corollary 2.1 reveals that the aggregate consumption function is not affected by the borrowing constraint, $\Sigma_i c_t^i = mpc_t X_t$. Clearing in the spot market for the consumption good then implies that aggregate wealth is $X_t = (mpc_t)^{-1} D_t$. Since the stock is the only asset in positive net supply, aggregate wealth must also equal the stock price. Thus, $S_t = (mpc_t)^{-1} D_t$: i.e. the stock price is the same in the

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[4]If beliefs are homogeneous, but ξ^S and/or ξ^B differs across investors, under logarithmic preferences there is insufficient heterogeneity to support trade and activate the constraints: the economy aggregates and the single-agent equilibrium prevails.

[5]This benchmark economy is the exchange version of the production economy in Detemple and Murthy (1994) with a dividend process set equal to the (endogenous) aggregate consumption process of the latter.

constrained and unconstrained economies with coefficients $\alpha^i_t = \beta + \hat{\mu}^i_t$, $\rho_{1t} = \sigma_t$, and $\rho_{2t} = 0$. While the price of a claim on aggregate consumption is invariant to the presence of constraints when all investors have logarithmic utilities and the same rate of time preference, the same is not true of the interest rate, for reasons that will be clear shortly. Hence, the market price of risk and prices of attainable claims (defined in section 4) will in general depend non-trivially on the constraints.

In the absence of borrowing or short sales restrictions (Corollary 2.1) the equilibrium interest rate is a wealth-weighted average of the interest rates which would prevail in homogeneous economies endowed with the beliefs of the diverse agents in the heterogeneous model: $r^u_t = \delta_t \hat{r}^1_t + (1-\delta_t)\hat{r}^2_t$ where $\hat{r}^i_t = \beta + \hat{\mu}^i_t - \sigma^2_t$, $i = 1,2$, and δ^u_t represents the equilibrium wealth share of agent 1. This is also the structure of the equilibrium interest rate in the economy with constraints (Theorem 2.2) when the constraints are inactive, i.e. when $(\Delta_t, \delta_t) \in L_0$, with the distribution of wealth given by (A.10) in the Appendix.

Now suppose that agent 1 is borrowing constrained and agent 2 is unconstrained. Then, in order to clear the market, the interest rate must induce agent 2 to willingly lend the amount that agent 1 wishes to borrow, $\xi_B X^1_t$, which is also the maximum that he/she is allowed to borrow (bond market clearing ensures that both agents cannot be borrowing constrained simultaneously). Since the unconstrained demand of agent 2 for lending $(1-\pi^2_t)X^2_t = [1-(\rho_t)^{-2}(\alpha^2_t-r_t)]X^2_t = [1-(\sigma_t)^{-2}(\hat{\mu}^2_t+\beta-r_t)]X^2_t$, the equilibrium interest rate must equal $r_t = \beta + \hat{\mu}^2_t - \sigma^2_t + \delta_t(1-\delta_t)^{-1}\sigma^2_t\xi_B$. Furthermore, this equilibrium configuration holds when agent 1's unconstrained demand for borrowing exceeds $\xi_B X^1_t$ and the constraint on short sales of the stock is not binding on agent 2, i.e. when $1-(\rho_t)^{-2}[\alpha^1_t-r_t] < -\xi_B$ and $(\rho_t)^{-2}[\alpha^2_t-r_t] \geq -\xi_S$. Evaluated at the equilibrium, $(\alpha^i_t = \beta + \hat{\mu}^i_t$, $\rho_{1t} = \sigma_t$, $\rho_{2t} = 0$, $r_t = \beta + \hat{\mu}^2_t - \sigma^2_t + \delta_t(1-\delta_t)^{-1}\sigma^2_t\xi_B)$, this yields the conditions $(1-\delta_t)^{-1}\xi_B < \Delta_t$ and $\delta_t \leq (1+\xi_S)/(1+\xi_S+\xi_B)$. Thus, agent 1 is borrowing constrained and agent 2 is unconstrained when $(\Delta_t, \delta_t) \in L_{1B}$. A similar analysis identifies the equilibrium interest rate when $(\Delta_t, \delta_t) \in L_{1S}$, L_{2B}, or L_{2S}.

It remains to examine events in which $(\Delta_t, \delta_t) \in L_{1B,2S}$ or $(\Delta_t, \delta_t) \in L_{1S,2B}$; correspondingly, $\delta_t = 1+\xi_S/(1+\xi_S+\xi_B)$ or $\delta_t = \xi_B/(1+\xi_S+\xi_B)$, and $\Delta_t > 1+\xi_S+\xi_B$ or $\Delta_t < -(1+\xi_S+\xi_B)$. The restriction to LCRL processes in Assumption 2.1 requires that sample paths of r be left continuous: $r_t = \lim_{s \uparrow t} r_s$. Hence,

$$r_t = \beta + \hat{\mu}^i_t + \sigma^2_t\xi_S \text{ if } (\Delta_t, \delta_t) \in L_{iB,jS} \text{ and } (\Delta_{t-}, \delta_{t-}) \in L_{iB},$$

for i,j = 1,2, i ≠ j, with agent i being borrowing constrained and agent j unconstrained. On the other hand,

$$r_t = \beta + \hat{\mu}^i_t - \sigma^2_t(1+\xi_B) \text{ if } (\Delta_t, \delta_t) \in L_{iB,jS} \text{ and } (\Delta_{t-}, \delta_{t-}) \in L_{jS},$$

with agent j being short sales constrained and agent i unconstrained. Since $\Delta_t \equiv (\hat{\mu}^1_t-\hat{\mu}^2_t)/\sigma^2 \neq 1+\xi_S+\xi_B$ the interest rate takes different values on $L_{iB,jS}$ depending on who is constrained. Thus, r fails to be right

11

continuous when the process (Δ,δ) crosses the half lines $L_{1B,2S}$ and $L_{2B,1S}$.[6] We have,

__Proposition 2.1__: For $i,j = 1,2$, $i \neq j$, consider the events in which $(\Delta_t,\delta_t) \in L_{iB,jS}$, and either $(\Delta_{t-},\delta_{t-}) \in L_{iB}$ with $(\Delta_{t+},\delta_{t+}) \in L_{jS}$, or $(\Delta_{t-},\delta_{t-}) \in L_{jS}$ with $(\Delta_{t+},\delta_{t+}) \in L_{iB}$. These are positive probability events in the equilibrium of Theorem 2.2. If $(\Delta_{t-},\delta_{t-}) \in L_{iB}$ (L_{jS}) and $(\Delta_{t+},\delta_{t+}) \in L_{jS}$ (L_{iB}) there is a positive (negative) jump in r_t of size $+$ $(-)$ $[\hat{\mu}_t^i - \hat{\mu}_t^j - \sigma_t^2(1+\xi_S+\xi_B)]$. The interest rate has continuous sample paths in all other events, P-a.s.

It is significant that jumps in r occur with positive probability despite the continuity of the information structure generated by the Brownian filtration $\mathfrak{I}_{(\cdot)}^i$. But note that the stock price process and its coefficients have continuous sample paths, as evidenced by (18) and (5)-(7).[7] One can reinterpret the equilibrium of Theorem 2.2 in terms of suitable "holding costs" to gain intuition for the likelihood of jumps. This would entail agent-specific deadweight costs that are non-zero for an individual agent only in events in which a constraint binds on him, yielding agent-specific returns.

3. A Categorization of Sources of Value

3.1 Speculative-value and Collateral-value Premia

In the context of a particular equilibrium, the consumption value of an asset's exogenously given payoffs to an investor may be measured using his IMRS. Define $\theta_{it}^i(\lambda_{it}^i) = \sigma_t^{-1} [\hat{\mu}_t^i + \beta + \lambda_{it}^i - r_t]$ where r is in (19), and the process λ_{it}^i in (15) evaluated at equilibrium prices is:

$$
\lambda_{it}^i \equiv \begin{cases}
0 & \text{for } (\Delta_t,\delta_t) \in L_0 \cup L_{jB} \cup L_{jS} \\
\hat{\mu}_t^i - \hat{\mu}_t^j + (X_t/X_t^j)\sigma_t^2\xi_B < 0 & \text{for } (\Delta_t,\delta_t) \in L_{iB} \\
\hat{\mu}_t^i - \hat{\mu}_t^j - (X_t/X_t^j)\sigma_t^2(1+\xi_S) > 0 & \text{for } (\Delta_t,\delta_t) \in L_{iS} \\
\lim_{s\uparrow t} \lambda_s^{1i} & \text{for } (\Delta_t,\delta_t) \in L_{iB,jS},
\end{cases}
\tag{22}
$$

for $i,j = 1,2$, $i \neq j$, where X^i/X solves (A.10) in the Appendix. In the equilibrium of Theorem 2.2 the IMRS of agent i between dates t and u, $0 \leq t \leq u \leq T$, is $m_{tu}^i \equiv (m_u^i/m_t^i)$ where

$$
m_t^i \equiv \exp[-\beta t]c_0^i/c_t^i = \exp[-\int_0^t [r_u + \gamma(\lambda_{1u}^i)]du]\eta_{\lambda t}^i,
\tag{23}
$$

---

[6]The proof of Theorem 2.2 in the Appendix shows that (Δ_t,δ_t) can enter $L_{iB,jS}$ with positive probability if and only if $(\Delta_{t-},\delta_{t-}) \in L_{iB}$ or L_{jS}, $i,j = 1,2$, $i \neq j$. Events in which $(\Delta_t,\delta_t) \in L_{iB,jS}$ and $(\Delta_{t-},\delta_{t-}) \in L_0$ have zero measure.

[7]Note that for $(\Delta_t,\delta_t) \in L_{iB,jS}$ every $r_t \in [\beta + \hat{\mu}_t^i + \sigma_t^2\xi_S, \beta + \hat{\mu}_t^j - \sigma_t^2(1+\xi_B)]$ is market-clearing. The LCRL restriction (Assumption 2.1) serves as a selection from this set. This assumption is innocuous and is made solely for expositional ease: alternative choices differ only on sets of zero Lebesgue measure and hence do not affect equilibrium allocations or the dynamics of any variable other than r.

$\eta_{\lambda}^i \equiv \exp[-\int_0^t \theta_{1s}^i(\lambda_{1s}^i)dv_{1s}^i - (1/2)\int_0^t (\theta_{1s}^i(\lambda_{1s}^i))^2 ds]$ is a $(P^i, \mathfrak{I}_{(\cdot)}^i)$-martingale, and $\gamma(\lambda_{1t}^i) \equiv \xi_S \text{Max}(\lambda_{1t}^i, 0) + (1+\xi_B)\text{Max}(-\lambda_{1t}^i, 0)$. The following proposition provides a decomposition of the stock's sources of value to an investor based on comparing the equilibrium stock price to his IMRS-based valuation of its dividends D.

<u>Proposition 3.1</u>: In the equilibrium of Theorem 2.2 the excess of the stock price over investor i's IMRS-based valuation of its dividends admits the decomposition:

$$S_t - E_t^i[\int_t^T m_{tu}^i D_u du] = E_t^i[\int_t^T m_{tu}^i \chi_u^i du] + E_t^i[\int_t^T m_{tu}^i \kappa_u^i du] \qquad (24)$$

for i = 1,2, t ∈ [0,T]. The first and second terms in the right hand side of (24) are the stock's "speculative-value premium" and "collateral-value premium," respectively, for investor i. They are associated with the "speculative gains" χ^i and "collateral services" κ^i that he imputes to the stock, defined as:

$$\chi_t^i \equiv \text{Max}(\lambda_{1t}^i, 0)S_t \qquad (25)$$

$$\kappa_t^i \equiv [\xi_S \text{Max}(\lambda_{1t}^i, 0) + \xi_B \text{Max}(-\lambda_{1t}^i, 0)]S_t \qquad (26)$$

where λ_{1t}^i in (22) is positive (negative) in events in which the constraint on short sales of the stock (leverage) is binding on investor i.

A review of the familiar "no expected gains from trade" result, or the martingale property of asset prices, is useful in understanding the preceding characterization of an asset's sources of value. (See Lucas (1978) and Duffie (1986) for discussions of this issue in frictionless economies). Recall that in a frictionless equilibrium every investor's IMRS or marginal utility of consumption-based valuation of every marketed asset's payoffs equals the price of the asset. For instance, in the "unconstrained" economy of Corollary 2.1, the stock price in terms of investor i's IMRS m_v^{iu}/m_t^{iu}, $0 \le t \le v \le T$, in equilibrium, may be expressed as $S_t \equiv E_t^i[\int_t^T (m_v^{iu}/m_t^{iu})D_v dv]$, i = 1,2. Stated differently, the stock price plus cumulative dividends corrected by i's IMRS is a martingale: $S_t m_t^{iu} + \int_0^t m_v^{iu} D_v dv = E_t^i[\int_0^T m_v^{iu} D_v dv]$, t ∈ [0,T]. Equivalently, there are no expected gains (capital gains plus dividends) from trade at equilibrium prices for investor i, i = 1,2, where the gains are "risk-adjusted" or corrected by his equilibrium IMRS. A further characterization is that, at the margin, every investor is indifferent to buying (a small quantity) of the stock at its equilibrium price even if he is obliged to maintain his holding forever: the stock price reflects only the consumption value of the payoffs derived from buy-and-hold strategies.

In contrast to the case of a frictionless equilibrium, (24) reveals that in the equilibrium of Theorem 2.2 the sum of the IMRS-adjusted stock price and cumulative dividends *plus* the process $\int_t^T m_u^i(\chi_u^i + \kappa_u^i)S_u du$ is a $(P^i, \mathfrak{I}_{(\cdot)}^i)$ martingale:

$$S_t m_t^i + \int_0^t m_u^i D_u du + \int_0^t m_u^i(\chi_u^i + \kappa_u^i) du = E_t^i[\int_0^T m_u^i D_u du] + E_t^i[\int_0^T m_u^i(\chi_u^i + \kappa_u^i) du],$$

for $t \in [0,T]$. Equivalently, substituting for χ^i and κ^i using (25)-(26) we have:

$$E_t^i[d(S_t m_t^i) + m_t^i D_t dt] = - m_t^i Max(\lambda_{1t}^i, 0) S_t dt - m_t^i[\xi_S Max(\lambda_{1t}^i, 0) + \xi_B Max(-\lambda_{1t}^i, 0)] S_t dt \qquad (27)$$

Similarly, it can be shown that a strategy of lending 1 unit of consumption at the riskless rate r accumulates expected IMRS-adjusted gains at the rate[8]

$$E_t^i[dm_t^i + m_t^i r_t dt] = - m_t^i Max(-\lambda_{1t}^i, 0) dt - m_t^i[\xi_B Max(-\lambda_{1t}^i, 0) + \xi_S Max(\lambda_{1t}^i, 0)] dt. \qquad (28)$$

Thus, there are expected gains from trade relative to investor i's IMRS, i = 1,2, since $\lambda_1^i \neq 0$ with positive probability. In particular, from (27)-(28), in events in which $\lambda_{1t}^i > 0$, investor i expects the strategy of selling 1 share of the stock and lending the proceeds at the riskless interest rate r to earn positive gains at the rate

$$E_t^i[S_t(dm_t^i + m_t^i r_t dt) - (d(m_t^i S_t) + m_t^i D_t dt)] = m_t^i \lambda_{1t}^i S_t dt \qquad (29)$$

in marginal utility terms, over and above capital gains and foregone dividends. Similarly, when $\lambda_{1t}^i < 0$, investor i expects the strategy of borrowing 1 unit of consumption at r and investing the proceeds in the stock to earn positive gains at the rate

$$E_t^i[(1/S_t)(d(m_t^i S_t) + m_t^i D_t dt) - (dm_t^i + m_t^i r_t dt)] = -m_t^i \lambda_{1t}^i dt \qquad (30)$$

in marginal utility terms, over and above capital gains and dividends.

Our characterization of the speculative-value and collateral-value premia in (24) may be understood by considering the benefits to investor i, at the margin, from purchasing an incremental quantity of the stock, say φ shares, at time $t \in [0,T)$ with his optimal consumption and portfolio policies being otherwise unchanged over [0,T]. If he simply holds the φ shares of stock through time T he would earn dividends which he values at $E_t^i[\int_t^T m_u^i D_u du]$ per share. But from (29) it follows that holding the φ shares when $\lambda_1^i > 0$ is inferior to selling them and lending the proceeds. This "timing" strategy enables investor i to realize positive gains at the rate $\lambda_{1u}^i S_u$ per share *in excess* of capital gains and foregone dividends by selling the stock purchased at t at the first time $u \in [t,T)$ that $\lambda_{1u}^i > 0$. Thus, the marginal benefits to investor i from buying a share of stock include the ability to realize "speculative gains" at the rate $\chi_t^i \equiv Max(\lambda_{1t}^i, 0) S_t$, $t \in [0,T)$; this accounts for the speculative-value premium $E_t^i[\int_t^T m_u^i \chi_u^i du]$ in (24). We term these gains speculative following HK (1978) who say that "investors exhibit speculative behavior if the right to resell a stock makes them willing to pay more for it than

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[8]To see this, compare the IMRS-adjusted returns $E_t^i[\int_t^\tau m_u^i r_u du + m_\tau^i]$ from a buy-and-hold lending strategy over $[t,\tau]$, $0 \leq t \leq \tau \leq T$, to the intial investment of 1 unit of consumption. Alternatively, compare the IMRS-based valuation of the terminal payoff $exp[\int_t^\tau r_u du]$ generated by a strategy of "rolling over" the initial investment of 1 unit from time t to τ. Proceeding as in the proof of Proposition 3.1, the corresponding IMRS-adjusted gains have a drift at the rate $-\gamma(\lambda_t^i) m_t^i$.

they would pay if obliged to hold it forever," and refer to the importance Keynes attached to this phenomenon.

The benefits to investor i, at the margin, from the purchase of the φ shares of stock also include collateral services. For instance, he not only has the right to resell the φ shares and invest the proceeds in the riskless asset, but in addition, he has the right to use the latter position as collateral against which he can *short sell* the stock. Alternatively, investor i can borrow using the φ shares of stock as collateral. In order to value these collateral services, which require determining when and which asset is short sold and the quantity of such short sales, three features are material.

First, investor i is subject to the constraints (10) on his overall portfolio. Second, note from (29)-(30) that investor i's right to short sell the stock (borrow) and invest the proceeds in the riskless asset (stock) has value only in events in which $\lambda_{1t}^i > (<) 0$. Lastly, recall that our marginal analysis of the purchase of the φ shares of the stock assumes that the investor's optimal consumption and portfolio policies are otherwise unchanged over [0,T]. This implies, in particular, that the constraint on short sales of the stock (leverage) binds on the rest of investor i's optimal portfolio precisely when $\lambda_{1t}^i > (<) 0$. Hence, in events in which $\lambda_{1t}^i > 0$, investor i has the right to short sell at most ξ_S shares of stock for every one of the φ shares purchased at the margin. It follows from (29) that an option to short sell ξ_S shares of stock enables him to realize gains at the rate $\xi_S Max(\lambda_{1t}^i,0)S_t$. Similarly, in events in which $\lambda_{1t}^i < 0$, investor i can borrow at most $\xi_B S_t$ per share of stock purchased at the margin, enabling him to realize gains at the rate $\xi_B Max(-\lambda_{1t}^i,0)S_t$, from (30). Hence, the marginal benefits to investor i from buying a share of stock include collateral services, or the ability to realize gains from short sales of the stock or borrowing, at the rate $\kappa_t^i \equiv [\xi_S Max(\lambda_{1t}^i,0) + \xi_B Max(-\lambda_{1t}^i,0)]S_t$ for which he is willing to pay the ex ante collateral-value premium $E_t^i[\int_t^T m_{tu}^i \kappa_u^i du]$ in (24).

Observe that in the limiting case of $\xi_S = 0$ and $\xi_B = \infty$ (no short sales of the stock are permitted and there is no constraint on borrowing), λ_1^i is positive in events in which the short sales constraint binds on investor i and is zero otherwise. Consequently, $\kappa^i = 0$: the stock has no value as collateral but continues to command a purely speculative-value premium. This special case of our model is similar to the no-short sales treatment of HK (1978) and AMP (1993). The qualitative difference between this paper and the latter work lies in the existence of collateral services when limited short sales are permitted, i.e. when $\xi_S > 0$ and/or $\xi_B > 0$. A categorization of the value derived from an investment in the riskless asset into consumption-value, speculative-value and collateral-value components may be obtained similarly; the details are omitted for the sake of brevity.[9] It follows that the value of any attainable payoff in the equilibrium of Theorem 2.2, being generated by a portfolio strategy in the stock and the riskless asset, has a similar decomposition.

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[9]This would require consideration of the strategies discussed in footnote 8. Also, note that an investment in the riskless asset has no speculative-value or collateral-value if $\xi_S = 0$ and $\xi_B = \infty$. The cases in which $\xi_S = \infty$ and $\xi_B = 0$, and $\xi_S = \xi_B = 0$ follow similarly.

3.2 Properties of Speculative Gains and Collateral Services

Our next proposition describes the properties of the speculative gains χ^i and collateral services κ^i imputed to the stock by investor $i = 1,2$. The proof follows from (25)-(26), straightforward differentiation using (15) and (22), and is hence not stated. A similar analysis applies to the corresponding gains and services imputed to an investment in the riskless asset. Recall that χ^i and κ^i are positive in events in which $\lambda_i^i > 0$ and $\lambda_i^i \neq 0$, respectively. We have:

Proposition 3.2:

(a) Let $(\Delta_t, \delta_t) \in L_{iS}$, or $(\Delta_t, \delta_t) \in L_{iS,jB}$ with $(\Delta_t, \delta_t) \in L_{iS}$. Then $\lambda_{it}^i > 0$. The speculative gain $\chi_t^i = \lambda_{it}^i S_t$ is linear and decreasing in ξ_S. It is a linear, increasing (decreasing) function of the stock price and the interest rate (his expectation of the stock's "instantaneous" rate of return and the stock's volatility). In equilibrium, χ_t^i is a decreasing function of agent i's expectation of the dividend growth rate and his share of aggregate wealth.

(b) Let $(\Delta_t, \delta_t) \in L_{iS}$ (L_{iB}), or $(\Delta_t, \delta_t) \in L_{iS,jB}$ $(L_{iB,jS})$ with $(\Delta_t, \delta_t) \in L_{iS}$ (L_{iB}). Then $\lambda_{it}^i > (<) 0$ and the collateral service $\kappa_t^i = \xi_S \lambda_{it}^i S_t$ $(-\xi_B \lambda_{it}^i S_t)$ is a convex, increasing-decreasing function of ξ_S (ξ_B).

The speculative gain from the stock is the product of the stock price and the shadow price for agent i associated with the constraint on short sales of the stock: $\chi_t^i = \max(\lambda_{it}^i,0)S_t$. The result in part (a) above follows since (15) implies $\chi_t^i = [-(\rho_t)^2\xi_S - \alpha_t^i + r_t)]S_t$ when $\lambda_{it}^i > 0$. Thus, the effects of ξ_S, α_t^i, r, etc., stem from their impact on the "tightness" of the constraint. The effects of agent i's expectation of the dividend growth rate and his share of aggregate wealth follow from evaluating χ_t^i using λ_{it}^i in (22). Note that when $\lambda_{it}^i > 0$, agent i is pessimistic relative to agent j and an increase in his share of the aggregate wealth decreases the interest rate and thus decreases λ_{it}^i.

When $\lambda_{it}^i > 0$, the collateral service that agent i imputes to the stock has an inverted U-shaped relationship with respect to increases in ξ_S (see Figures 2 and 3). To understand the source of this interesting result note that when $\lambda_{it}^i > 0$, $\partial \kappa_t^i / \partial \xi_S = [\lambda_{it}^i + (\partial \lambda_{it}^i / \partial \xi_S)\xi_S]S_t$ where $\partial \lambda_{it}^i / \partial \xi_S < 0$ from (22). When no short sales of the stock are permitted (i.e. $\xi_S = 0$), clearly collateral has no value. As limited short sales are allowed and this constraint is progressively relaxed (i.e. ξ_S is increased), every asset must have increasing collateral value to investor i: $\lambda_{it}^i > -(\partial \lambda_{it}^i / \partial \xi_S)\xi_S$. However, the latter effect cannot hold for all ξ_S since the shadow price associated with the short sales constraint on agent i, λ_{it}^i, is decreasing in ξ_S. Thus, for a sufficiently large relaxation of the margin requirement, the decreased value to investor i of selling short the stock offsets his increased ability to do so; the net effect is one of lower collateral services imputed to the stock. Finally, for a "loose-enough" margin requirement on short sales, the constraint will cease to bind on investor i at which

point collateral has no value to him. The intuition behind the inverted U-shaped relationship of κ_t^i with respect to ξ_B when $\lambda_{1t}^i < 0$ is similar. Also, the effects on κ_t^i of changes in other variables is straightforwardly determined from their effects on χ_t^i.

<div align="center">INSERT FIGURES 2 AND 3 HERE</div>

Combining the results of parts (a) and (b) of Proposition 3.2, it is interesting to note that agent i's expected gains from trade in the stock, which equal the sum of the speculative gains and the collateral services $\chi_t^i + \kappa_t^i$, are not necessarily decreasing in ξ_s. This yields the empirical implication that the deviation of an asset's return from an IMRS-based prediction can increase as a short sales constraint is relaxed. The reason is understood as follows. Consider events in which $\lambda_{1t}^i > 0$ for instance: the constraint on short sales of stock binds on agent i. While relaxing the margin requirement on short sales of the stock certainly reduces an investor's expectation of the speculative gains from selling every share (since $\partial \lambda_{1t}^i / \partial \xi_s < 0$), it also has the effect of increasing the number of shares that can be sold short. In other words, as depicted in Figure 2, a higher level of the collateral services κ_t^i can offset the decrease in the speculative gains χ_t^i so that agent i's expected gains from trade in the stock, $\chi_t^i + \kappa_t^i$, increases in ξ_s. Figure 3 illustrates the alternative case in which the expected gains from trade decrease as the margin requirement is relaxed.

3.3 Further Discussion of Related Work

While it is generally acknowledged that the excess of an asset's price over its "fundamental value" is a "bubble," the notion that a given payoff has an unambiguous fundamental value is clearly problematic. The measures of fundamental value discussed in the literature on bubbles include both valuations of an asset's dividends based on investors' equilibrium marginal utility of consumption operators (IMRSs), and alternatively, valuations based on state prices. Based on the former notion of fundamental value, AMP (1993) define and discuss "expected bubbles."[10] Relative to this measure, the speculative-value and collateral-value components in an asset's price that we have discussed may well be interpreted as bubbles.

The alternative notion that an asset's fundamental value is based on state prices is used by Santos and Woodford (1996), for instance. Observe that the stock price in (18) may be written as the present value of its dividends in terms of the state price density: $S_t = E_t^i[\int_t^T D_u(\eta_u^i/\eta_t^i)\exp\{-\int_t^u r_s ds\}du]$. Equivalently, the stock price

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[10]AMP (1993) provide an extensive discussion of the problematic notion of an unambiguous fundamental value. They also establish that there are equilibria with "strong" bubbles in which an asset's price exceeds an upper bound on *any* reasonable notion of fundamental value; such bubbles require asymmetric information and the lack of common knowledge about agents' trades, in addition to a short sales constraint binding on every agent.

plus cumulative dividends corrected by η^i and discounted at the riskfree rate is a martingale: $S_t\eta^i_t\exp\{-\int_0^t r_s ds\}$ $+ \int_0^t D_u\eta^i_u\exp\{-\int_0^u r_s ds\}du = E^i_t[\int_0^T D_u\eta^i_u\exp\{-\int_0^u r_s ds\}du]$. Thus, there are no expected gains from trade relative to the EMM Q: $E^i_t[d(S_t\eta^i_t\exp\{-\int_0^t r_s ds\}) + D_t\eta^i_t\exp\{-\int_0^t r_s ds\}dt] = 0$; i.e. there is no bubble in the stock price relative to its fundamental value based on state prices.[11]

As this discussion illustrates, the two measures of fundamental value need not coincide for any investor in the presence of frictions. In fact, while the fundamental value based on state prices subsumes the consumption value of dividends, it also reflects other sources of value that investors derive from holding the (tradable) asset. Thus, there is no inconsistency between the absence of a bubble relative to a fundamental value based on state prices on the one hand, and speculative behavior on the other.

Our result on the existence of a collateral-value component bears similarities to some results of Hindy (1995). While, in general, there is a relationship between the structures of constraints in our paper and Hindy if there are no more than 2 assets (as in sections 1-3 but unlike sections 4-5 of this paper), there are significant differences in our respective papers as discussed in section 1.[12] Our demonstration of an equilibrium with binding constraints underscores the idiosyncratic nature of the value to collateral and yields further insights into the determinants of collateral-value premia. Also note that in Hindy, unlike in Harrison and Kreps (1979), there is no probability measure under which *all* assets' prices are martingales. Instead, normalized prices of securities with high (low) collateralizability are martingales (submartingales) under a change of probability measure. In contrast, there necessarily exists an EMM in our model since we do not allow for the existence of multiple assets with the same cashflows but different prices.

4. No-Arbitrage Pricing and Financial Innovation

In this section we examine the implications of portfolio constraints for no-arbitrage or replication-based pricing and, accordingly, start with considering payoffs that are attainable in the equilibrium of Theorem 2.2.

These are payoffs that investors can synthesize by transacting at the equilibrium prices in the stock and riskless asset markets subject to the constraints on short sales of stock and leverage. Hence, we say that a $\mathfrak{I}_T^{D,Y}$-measurable, square-integrable payoff G is *attainable* if it equals the time T value of a portfolio strategy defined by the $\mathfrak{I}^i_{(.)}$-progressively measurable weight in the stock $\phi_G \in \mathfrak{L}^2[0,T]$ that satisfies $\phi_G \geq -\xi_S$ and $1-\phi_G \geq -\xi_B$, and constructed at prices (18)-(19) starting with some initial capital g > 0 at t = 0. This attainable (or

synthetic) payoff has the following representation with respect to agent i = 1,2's beliefs:

$$G \equiv g \, \exp\{\int_0^T [\phi_{Gt}(\hat{\mu}_t^i+\beta-r_t)+r_t-\tfrac{1}{2}(\phi_{Gt}\sigma_t)^2]dt + \int_0^T \phi_{Gt}\sigma_t dv_{it}^i\}, \tag{31}$$

Its no-arbitrage based price relative to the EMM Q in the equilibrium of Theorem 2.2 is:

$$E^Q[\exp\{-\int_t^T r_u du\}G \mid \mathfrak{I}_t^{D,Y}] = g \, \exp\{\int_0^t \phi_{Gu}\sigma_u dv_{iu}^i + \int_0^t [\phi_{Gu}(\hat{\mu}_u^i+\beta-r_u)+r_u-\tfrac{1}{2}(\phi_{Gu}\sigma_u)^2]du\}, \; t \in [0,T], \tag{32}$$

which equals the value of the portfolio that replicates G. One can also allow for attainable "flow" payoffs over a time interval. Also note that one can use the EMM Q to arrive at no-arbitrage based bounds on the prices of payoffs which are not attainable but which can be superreplicated, i.e. dominated, by admissible trading strategies.

Now consider the introduction of a market in a new, "real" security with a payoff G at time T given by (31). The price of this derivative security would be given by no-arbitrage relative to the EMM Q if and only if the new security market were redundant. The latter would appear to be the case since investors can synthesize the payoff G prior to the financial innovation using an admissible trading strategy at the prices (18)-(19). However, as we will see, the new security may not be redundant even if only one of the two agents i = 1,2 finds the constraint binding in the equilibrium of Theorem 2.2.

If in fact the introduction of the new market has no impact, investors are able to trade the derivative at the no-arbitrage price in (32). At the prices (18)-(19) and (32) an admissible consumption-portfolio policy (c^i,π_S^i,π_B^i) for agent i = 1,2 is such that $(c^i,\pi_S^i,\pi_B^i) \in \mathfrak{L}_+[0,T]x(\mathfrak{L}^2[0,T])^2$, is $\mathfrak{I}_{(t)}^i$-progressively measurable, satisfies the portfolio constraints, $\pi_S^i \geq -\xi_S$ and $\pi_B^i \geq -\xi_B$, and the nonnegativity restriction $X_t^i \geq 0$, where X^i solves:

$$dX_t^i = X_t^i \pi_{Bt}^i r_t dt - c_t^i dt + X_t^i \pi_{St}^i [(\hat{\mu}_t^i+\beta)dt+\sigma_t dv_{it}^i] + X_t^i(1-\pi_{St}^i-\pi_{Bt}^i)\{[\phi_{Gt}(\hat{\mu}_t^i+\beta-r_t)+r_t]dt+\phi_{Gt}\sigma_t dv_{it}^i\} \tag{33}$$

for t ∈ [0,T] with $X_0^i = x^iS_0$. An optimal consumption-portfolio policy (c^i,π_S^i,π_B^i) is an admissible policy such that there is no other admissible policy $(c^{i'},\pi_S^{i'},\pi_B^{i'})$ with $U^i(c^{i'}) > U^i(c^i)$. The equilibrium of Theorem 2.2 is a rational expectations equilibrium of the economy with the derivative market if there exists an optimal (c^i,π_S^i,π_B^i), with c^i in (21), i = 1,2, and all markets clear: $\Sigma_i c^i = D$, $\Sigma_i \pi_S^i X^i = S$, $\Sigma_i \pi_B^i X^i = 0$, and $\Sigma_i(1-\pi_S^i-\pi_B^i)X^i = 0$. We have:

Proposition 4.1: Consider the economy of section 1 with an additional market in a derivative whose date T payoff G in (31) satisfies either

$$\phi_{Gt} \neq 0 \text{ in some event in which } \sigma^{-2}(\hat{\mu}^i+\beta-r) < -\xi_S,$$

$$\text{or,} \quad \phi_{Gt} \neq 1 \text{ in some event in which } \sigma^{-2}(\hat{\mu}^i+\beta-r) > 1+\xi_B, \tag{34}$$

for i = 1 or 2, t ∈ [0,T]. Then the equilibrium of Theorem 2.2 is no longer valid: the derivative market is not

redundant.[13] The real security with payoff G, unlike its synthetic counterpart, is not priced by no-arbitrage relative to the EMM Q.

Observe that condition (34) requires the derivative's replicating portfolio in the equilibrium of Theorem 2.2 to have a non-zero weight in the stock (riskless asset) in some subset of the events in which the constraint on short sales of the stock (leverage) binds on either investor $i = 1$ or 2. Hence, at the prices (18)-(19) and (32), at least one agent can use the derivative to circumvent the short sales and leverage constraints in some events either completely or partially. At these revised optimal policies the prices in (18)-(19) and (32) do not clear markets: the equilibrium of Theorem 2.2 is no longer valid. Note that this result would clearly hold even if the constraints were binding on only one of the agents in the equilibrium of Theorem 2.2. Also observe that while we have assumed there are no separate constraints on positions in the derivative, this is not essential. It serves as a simple way of capturing the often cited reason why derivatives are attractive to investors: they permit strategies that would otherwise have been infeasible for reasons of regulatory constraints. It is straightforward to see that the equilibrium of Theorem 2.2 is no longer valid even when there are margin requirements in the derivative market.

Proposition 4.1 shows that it may not be appropriate to price derivatives by no-arbitrage even if the primitive assets' prices and prevailing constraints on trading strategies suggest that they can be replicated by admissible portfolios. The reason is that real securities differ from their synthetic counterparts in the presence of portfolio constraints. Such differences, we argue, affect interpretations of the results of a recent literature which addresses no-arbitrage based valuation in the presence of short sales or leverage constraints. In particular, both pricing bounds results for unattainable claims and the single prices to which such bounds collapse if the claims are attainable, apply to synthetic payoffs and not real securities.

In general, the effects of constraints on asset positions can be similar to those of constraints on agents' information (Milne and Shefrin (1987)). The result in Proposition 4.1 is similar to that of Grossman (1988) and Back (1993) who show that despite investors' ability to synthesize the payoff of an option, the introduction of a market in the option can alter any underlying asymmetry of information, and thereby have an equilibrium impact. Our result differs from that of Heath and Jarrow (1987) who consider the implications of incorporating margin requirements in a Black and Scholes economy (1973) because our general equilibrium analysis takes account of the impact of the revision of investors' optimal choices on the prices of the underlying, primitive assets. Chen (1995) shows that IMRS-based values of attainable payoffs may differ from prices implied by

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[13]It is not straightforward to verify that the unconstrained equilibrium obtains. This entails verifying a condition involving the derivative's endogenous return volatility coefficients, which cannot be solved analytically.

no-arbitrage. A principal difference in our respective demonstrations of the failure of pricing by no-arbitrage is that Chen considers "small" payoffs which are marketed to the investor(s) with the highest IMRS-based marginal valuation through the use of the "rational conjecture condition" (see Allen and Gale (1988)). We consider the introduction of markets in attainable payoffs; while no-arbitrage may fail to price such payoffs, so might the existing equilibrium IMRSs of investors.

It is possible to extend our model to the case in which constraints bind on some but not all primitive assets, unlike the equilibrium of Theorem 2.2, to show that while pricing by no-arbitrage fails for some attainable payoffs, it is valid for others. The latter class of payoffs are those that are synthesized by assets not subject to any constraints. Furthermore, it can be proved that such payoffs cannot be valued with knowledge of only an individual's IMRS; instead, knowledge of all investors' IMRSs is necessary. This result is important since it outlines circumstances in which the ease of pricing by no-arbitrage is underscored, relying as it does on knowledge of only the market prices of the underlying primitive assets.

5. Extensions

We now consider an extension of our model to an economy with multiple stocks and more than two investors with diverse utility functions. We consider the case of homogeneous beliefs to also illustrate that heterogeneity in beliefs is not necessary to our arguments.[14] There are $i = 1,...,I$ agents with common information $\mathfrak{I}_{()}^W$ generated by a (dx1) Brownian Motion process W. Stock $n = 1,...,N$, $1 \leq N \leq d$, is a claim on dividends D_n which satisfies an analogue of (2) with drift μ_n and volatility $\sigma_n \equiv (\sigma_{n1},...,\sigma_{nd})$. We take as given cum-dividend stock returns and a cumulative riskless rate of return R with Ito representations,

$$dS_{nt} + D_{nt}dt = S_{nt}\alpha_{nt}dt + S_{nt}\rho_{nt}dW_t, \quad t \in [0,T]; \quad S_{n0} = \bar{S}_n, \tag{35}$$

$$dR_t = r_t dt, \quad t \in [0,T]; \quad R_0 = 0. \tag{36}$$

Agent i has preferences given by $U^i(c^i) = E[\int_0^T e^{-\beta t} u^i(c_t^i)dt]$, and is endowed with $x_n^i > 0$ shares of stock n. The utility functions $u^i(.)$ and the $\mathfrak{I}_{()}^W$-progressively measurable processes α_n, $\rho_n \equiv (\rho_{n1},...,\rho_{nd})$, r, μ_n, and σ_n, $n = 1,...,N$, satisfy standard assumptions. Let $\pi^i \equiv (\pi_1^i,...,\pi_N^i)$ be the vector of portfolio weights of agent i. The portfolio constraints are

$$\pi^i \in K \equiv \{\pi_n \geq -\xi_n, \xi_n \in [0,\infty), n = 1,...N, \text{ and } 1-\sum_{n=1}^N \pi_n \geq -\xi_B, \xi_B \in [0,\infty)\}. \tag{37}$$

Admissible consumption-portfolio strategies which satisfy (37), individual optimality, and equilibrium are defined as in section 1.

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[14]In the interest of space only the salient features of the economy are described. It is essentially a continuous time version of Lucas (1978) modified to include diverse preferences and portfolio constraints.

For i = 1,...,I, assume that there exists a solution (c^{i*}, π^{i*}) to agent i's optimization problem at the prices (35)-(36) such that for every $t \in [0,T]$ he finds the constraint associated with some asset binding at some $u \in [t,T]$ with positive probability. Then, from Cvitanic and Karatzas (1992), there exist $\mathfrak{I}^W_{(\cdot)}$-progressively measurable multipliers $\lambda^i \equiv (\lambda^{i\top}_S, \lambda^{i\top}_Z)^\top$ and $\gamma(\lambda^i) \equiv \sup\{-\pi^i \lambda^i_S | \pi^i \in K\}$ which characterize optimality for agent i, where $\lambda^i_S \equiv (\lambda^i_1,...,\lambda^i_N)^\top$, $\lambda^i_Z \equiv (\lambda^i_{N+1},...,\lambda^i_d)^\top$, $E[\int_0^T \|\lambda^i_t\|^2 dt] < \infty$ and $E[\int_0^T \gamma(\lambda^i_t) dt] < \infty$. Also assume that the prices (35)-(36) and allocations $\{(c^{i*}, \pi^{i*})\}$ are an equilibrium. Let $C \equiv \sum_{n=1}^N D_n$ denote aggregate consumption whose growth rate has drift $\mu_C \equiv \sum_{n=1}^N \mu_n D_n/C$ and volatility $\sigma_C \equiv \sum_{n=1}^N \sigma_n^\top D_n/C$. Define $\alpha \equiv (\alpha_1,...,\alpha_N)^\top$, $\rho \equiv (\rho_1^\top,...,\rho_N^\top)^\top$, and $V \equiv [\rho^\top, \rho_Z^\top]^\top$ where ρ_Z is an arbitrary (d-N)xd matrix such that V is of full rank. Also let $u^i_c(.)$, $u^i_{cc}(.)$, $u^i_{ccc}(.)$ denote the first, second and third derivatives of $u^i(.)$. Define $\bar{A}^i_t \equiv -u^i_c(c^i_t)/u^i_{cc}(c^i_t)$, $\bar{B}^i_t \equiv -u^i_{ccc}(c^i_t)(\bar{A}^i_t)^2/u^i_{cc}(c^i_t)$, $\bar{A}_t \equiv \sum_i \bar{A}^i_t$, $\bar{B}_t \equiv \sum_i \bar{B}^i_t$, $w^i_{At} \equiv \bar{A}^i_t/\bar{A}_t$, and $w^i_{Bt} \equiv \bar{B}^i_t/\bar{B}_t$. With this notation, we have,

Proposition 5.1: Consider the economy with multiple stocks and more than two agents described above. The components of $\lambda^i_S \equiv (\lambda^i_1,...,\lambda^i_N)^\top$, i = 1,...,I, have the following characterization for assets n ≠ p:

$$\lambda^i_{nt} = 0 \text{ if } \pi^{i*}_{nt} > -\xi_n \text{ and } 1-\sum_{n=1}^N \pi^{i*}_{nt} > -\xi_B$$

$$\lambda^i_{nt} > 0 \text{ if at time t agent i is short sales constrained in n and } 1-\sum_{n=1}^N \pi^{i*}_{nt} > -\xi_B$$

$$\lambda^i_{nt} = \lambda^i_{pt} < 0, \text{ if } \pi^{i*}_{nt} > -\xi_n, \pi^{i*}_{pt} > -\xi_p, \text{ and the leverage constraint binds at t, and,} \qquad (38)$$

$$\lambda^i_{nt} \gtreqless 0 \text{ and } \lambda^i_{nt} > \lambda^i_{pt} \text{ if at time t agent i is both short sales constrained in n and}$$

$$\text{leverage constrained, and } \pi^{i*}_{pt} > -\xi_p.$$

The equilibrium price of stock n = 1,...,N admits the decomposition:

$$S_{nt} = E_t[\int_t^T m^i_{tu}(D_{nu} + \chi^i_{nu} + \kappa^i_{nu}) du], \ t \in [0,T], \qquad (39)$$

into consumption-value, speculative-value, and collateral-value components, in terms of investor i's IMRS m^i. The speculative gains and collateral services that agent i imputes to stock n = 1,...,N are,

$$\chi^i_{nt} \equiv (\lambda^i_{nt} - \min\{0,\lambda^i_{1t},...,\lambda^i_{Nt}\})S_{nt} \geq 0, \quad \text{and,} \qquad (40)$$

$$\kappa^i_{nt} \equiv (\gamma(\lambda^i_t)+\min\{0,\lambda^i_{1t},...,\lambda^i_{Nt}\})S_{nt} \geq 0, \qquad (41)$$

where $\qquad \gamma(\lambda^i) \equiv \sum_{n=1}^N \xi_n \lambda^i_n + [1+\sum_{n=1}^N \xi_n+\xi_B]\max\{\lambda^{i-}_1,...,\lambda^{i-}_N\}$

and $\lambda^i_{nt} > \min\{0,\lambda^i_{1t},...,\lambda^i_{Nt}\}$ if and only if agent i is short sales constrained in asset n at date $t \in [0,T]$. The equilibrium interest rate and the risk premia of the stocks are respectively given by, for $t \in [0,T]$,

$$r_t = \beta + (\bar{A}_t)^{-1}C_t\mu_{Ct} - \tfrac{1}{2}\bar{B}_t(\bar{A}_t)^{-3}C_t^2\sigma_{Ct}^\top\sigma_{Ct} - \sum_i w^i_{At}\gamma(\lambda^i_t) + \bar{B}_t(\bar{A}_t)^{-2}C_t\sigma_{Ct}^\top V_t^{-1}(\sum_i w^i_{At}\lambda^i_t - \sum_i w^i_{Bt}\lambda^i_t) \qquad (42)$$

$$- \tfrac{1}{2}\bar{B}_t(\bar{A}_t)^{-1}[(\sum_i w^i_{At}\lambda^i_t)^\top(V_tV_t^\top)^{-1}(\sum_i w^i_{At}\lambda^i_t) - 2(\sum_i w^i_{At}\lambda^i_t)^\top(V_tV_t^\top)^{-1}(\sum_i w^i_{Bt}\lambda^i_t) + \sum_i w^i_{Bt}\lambda^{i\top}_t(V_tV_t^\top)^{-1}\lambda^i_t],$$

and $\qquad\qquad \alpha_t - r_t 1_N = (\bar{A}_t)^{-1}C_t\rho_t\sigma_{Ct} - \sum_i w^i_{At}\lambda^i_{St}. \qquad (43)$

The speculative behavior evidenced by Proposition 5.1 is richer than in the two asset case of Proposition 3.1. To see this, first observe that as in section 3, the IMRS-adjusted gains process for stock n = 1,...,N implied by (39), $\int_t^T [d(S_{nu}m_u^i) + m_u^i D_{nu} du]$, has expected value $- E_t \int_t^T [\lambda_{nu}^i + \gamma(\lambda_u^i)] S_u m_u^i du$, and that for a unit investment in the riskless asset has a drift at the rate $-\gamma(\lambda_t^i) m_t^i$. Next, arguing as in sub-section 3.1, ownership of a share of stock n, at the margin, allows investor i to sell the share and acquire any other stock $p \neq n$ or the riskless asset. Hence, the option of selling stock n to acquire stock p has expected gains at the rate $(\lambda_{nt}^i - \lambda_{pt}^i)^+ S_{nt}$. Similarly, the option of selling stock n and lending the proceeds has expected gains at the rate $(\lambda_{nt}^i)^+ S_{nt}$. It follows that a speculative strategy in this setting with more than 2 assets must involve both timing and *selection*: if it pays investor i to sell the share of stock n he may do better by investing the sale proceeds in stock p_1 rather than in the riskless asset or some other stock p_2. Thus, the speculative gains χ_{nt}^i are given by $\max[\max_p(\lambda_{nt}^i - \lambda_{pt}^i)^+, (\lambda_{nt}^i)^+] S_{nt}$ and may be rewritten as in (40); they are positive in a given event if and only if investor i is short sales constrained in stock n in this event. These gains are realized by a strategy of both the timely sale of the share of stock n and from a judicious choice of which other asset to acquire.

The collateral services provided by an individual asset depend upon all assets in which an investor is constrained. The option of borrowing (short selling) 1 unit worth of the riskless asset (some stock p) and investing the proceeds in stock n = 1,...,N has IMRS-adjusted expected gains at the rate $(-\lambda_{nt}^i)^+ ((\lambda_{pt}^i - \lambda_{nt}^i)^+)$. From (38) it is clear that the gains from such borrowing (short sales) are maximized by investing the proceeds in a stock k such that $\min\{0, \lambda_{1t}^i, ..., \lambda_{Nt}^i\} = \lambda_{kt}^i$, i.e. a stock in which investor i is unconstrained, if such a stock exists. Alternatively, if agent i is short sales constrained in all stocks he must be unconstrained in the riskless asset; in this case short selling 1 unit worth of stock n = 1,...,N and lending the proceeds at the riskless rate has expected gains at the rate $\lambda_{nt}^i > 0$. From (38), the option to borrow (short sell stock p) has value in a given event if and only if the constraint on leverage (short sales of stock p) binds on investor i in the same event. Hence, it follows as in sub-section 3.1, that in these events the benefits at the margin from a share of stock n worth S_{nt} include the ability to borrow $\xi_B S_{nt}$ at the riskless rate and to short sell $\xi_p S_{nt}$ worth of stock p = 1,...,N. Thus the collateral service to agent i is $\kappa_{nt}^i \equiv [-\xi_B \min\{0, \lambda_{1t}^i, ..., \lambda_{Nt}^i\} + \sum_{p=1}^N \xi_p(\lambda_{pt}^i - \min\{0, \lambda_{1t}^i, ..., \lambda_{Nt}^i\})] S_{nt}$, which may be rewritten as in (41). Observe that in the no-short-sales and no-borrowing case ($\xi_B = \xi_n = 0 \; \forall$ n), $\kappa_{nt}^i = 0 \; \forall$ i,n. In contrast, with limited short sales, every asset has a non-trivial collateral-value component for every agent if he is constrained in some asset.

Thus, investors' marginal valuations of assets in an equilibrium with binding short sales constraints are not based solely on the payoffs corresponding to buy-and-hold strategies. Instead, they correspond to portfolio strategies that maximize their respective expected gains from trade. This follows from considering the expected gains for a portfolio policy $\pi^i \in K$, $-E_t^i \int_t^T (\sum_{n=1}^N \pi_{nu}^i \lambda_{nu}^i + \gamma(\lambda_u^i)) X_u^i m_u^i du$, and noting that agent i's optimal portfolio $\pi^{i*} \in$ argmax $-[\sum_{n=1}^N \pi_n^i \lambda_n^i + \gamma(\lambda^i)]$ subject to $\pi^i \in K$.

The results (42)-(43) on the interest rate and risk premia are consistent with the literature on aggregate consumption-based asset pricing relationships under frictions (see Luttmer (1996) and He and Modest (1995), for instance). Risk premia in (43) are only partially determined by the instantaneous covariation $\rho_t \sigma_{Ct}$ of asset returns with aggregate consumption. The last term in (43) is the deviation from the consumption-based CAPM (CCAPM) of Breeden (1979) due to binding constraints and is equal to a weighted average of the λ^i's across investors where the weights are given by their absolute risk tolerances relative to that of the aggregate. For instance, when at least one investor is simultaneously leverage constrained and short sales constrained in a stock n, the CCAPM estimate may be greater or less than the risk premium α_{nt} - r. These results are similar to ones obtained by Cuoco (1997).[15] Observe that the deviation of α_{nt} - α_{pt} from the CCAPM prediction is $\Sigma_i w^i_{At}[\chi^i_{pt}/S_{pt} - \chi^i_{nt}/S_{nt}]$: relative mispricing across assets n $\neq$ p may be attributed to their differing speculative gains. Turning to the interest rate, note that the first three of the eight terms in (42) capture the usual effects of time preference and aggregate consumption. The remaining terms account for the presence of the constraints (37) and the incompleteness of markets when N < d. The fourth, sixth, and eighth terms are negative (if $\bar{B}^i_t$ > 0: convex marginal utility) and contribute to a reduction in the interest rate. A priori, the fifth and seventh terms could be positive and, in principle, could offset the aforementioned reduction.

The characterizations above of equilibrium prices continue to hold when investors also have heterogeneous beliefs. Assume that the structure of beliefs of agent i = 1,...,I is described by his prior P^i and the (dx1) innovation process v^i along the lines of the model in section 2. Let $\Delta^i \equiv v^i - v^1$, i = 2,...,I, now denote the difference in beliefs of agent i relative to that of the reference agent i = 1. Then (42) and (43) hold with λ^i_S replaced by $\lambda^i_S - \rho\Delta^i$, as a correction for the heterogeneity in beliefs. The representations (39)-(41) of prices in terms of consumption-value, speculative-value and collateral-value components remain valid. Similarly, one can also extend Proposition 5.1 to the case where the constraints are agent-specific.

6. Concluding Remarks.

The general equilibrium model with binding portfolio constraints extends straightforwardly to i = 1,...,I > 2 agents with logarithmic utilities and heterogeneous beliefs. In this case, the number of possible configurations in (17) increases to 3^I - 2, and the dimension of the equilibrium "state variable" increases to 2(I-1). Similarly, one can also allow for agent-specific constraints with ξ_S and ξ_B differing across investors. The results of the two agent equilibrium carry over, in a qualitative sense, and imply that the individual-specific deviations of

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[15]Cuoco (1997) investigates the effects of constraining dollar investments in assets to lie in a closed, convex cone in the presence of stochastic labor income. Also similar to Cuoco, note that deviations of risk premia from the CCAPM in (43) may be identical across assets: when some investors are unconstrained and all others are leverage constrained.

asset returns from IMRS-based predictions, or errors relative to Euler equalities, are correlated. Our closed form solutions in principle permit the estimation of the degree to which constraints bind. For a given dispersion in beliefs, $\{\hat{\mu}^i - \hat{\mu}^1\}_{i \neq 1}$, the likelihood of binding constraints increases as the asset return volatility decreases because of the increased absolute demands for assets by investors. On the other hand, periods of high return volatility may also witness an increased likelihood of binding constraints if they are associated with an increased dispersion in beliefs. A further equilibrium implication, as can be inferred from Figure 1, is that a less-wealthy agent is more likely to find a portfolio weight constraint binding. In this sense, the equilibrium predictions here may be similar to those suggested by considerations of liquidity constraints in models with labor income. However, note that by the same token, a less wealthy agent also has a lower impact on return deviations from predictions using IMRSs based on aggregate consumption. Our model can also be extended to study the effects of binding constraints on the term structure of interest rates. This is a potentially useful application since the results in this paper show that, locally, a tightening of the constraint on leverage (short sales of stock) decreases (increases) the interest rate and increases (decreases) the market price of risk.

The main result on the existence of speculative-value and collateral-value premia in prices was also demonstrated under diversity in utility functions. The presence of a collateral-value component in asset prices indicates that all assets may individually experience deviations from IMRS-based pricing including assets in which trade is unrestricted. This collateral component becomes relevant under very weak conditions, namely whenever some subset of assets is subject to proportional short sales constraints and one of these constraints binds on an individual with positive probability. However, such deviations of returns are the same across unconstrained assets.

It also bears emphasizing that in our model the right to resell has value at the margin. Assets with the same dividends which may not be retraded have, at the margin, lower values than their tradable counterparts. Investors value the ability to re-trade assets because it enables speculation. Furthermore, the speculative premium leads to an increased value for the asset as collateral. These properties suggest different incentive effects for intermediaries engaged in financial innovation, for instance in the context of private placement versus exchange listing.

Appendix: Proofs.

Proof of Theorem 2.1: We solve the individual consumption-portfolio optimization problem with the portfolio constraints and incomplete markets by embedding it in a larger family of "unconstrained" optimization problems that correspond to artificial economies described below. The details of the theory that we employ can be found in Cvitanic and Karatzas (1992).

Let **H** denote the Hilbert space of processes $\lambda \equiv (\lambda_1,\lambda_2)$ that are $\mathfrak{I}^i_{(\cdot)}$-progressively measurable, with values in $\mathbf{R}^2$, and such that $E^i[\int_0^T\|\lambda_t\|^2dt]<\infty$. Let $K \equiv \{\pi \in \mathbf{R}^2: \pi_1 \in [-\xi_S,1+\xi_B], \pi_2=0\}$ where $\xi_S \in [0,\infty]$ and $\xi_B \in [0,\infty]$. Define the map $\gamma(\lambda) \equiv \sup_{\pi \in K}(-\pi\lambda)$ which represents the support function of the set -K and let **D** $\equiv \{\lambda \in \mathbf{H}: E^i[\int_0^T\gamma(\lambda_t)dt] < \infty\}$. For $\lambda \in \mathbf{D}$ consider the artificial economy which is identical in all respects to the economy of section 1 except for its modified, private investment opportunity sets,

$$dS_t+D_tdt = S_t[[\alpha^i_t+\gamma(\lambda_t)+\lambda_{1t}]dt + \rho_{1t}dv^i_{1t} + \rho_{2t}dv^i_{2t}], \ t \in [0,T]; \ S_0 \text{ given}, \tag{A.1}$$

$$dR_t = [r_t+\gamma(\lambda_t)]dt, \ t \in [0,T]; \ R_0 \equiv 1, \tag{A.2}$$

$$dZ^i_t = Z^i_t[\gamma(\lambda_t)+\lambda_{2t}]dt + Z^i_t\rho^Z_tdv^i_{2t}, \ t \in [0,T]; \ Z^i_0 \text{ given}. \tag{A.3}$$

In this artificial economy a triplet (c^i,π^i_1,π^i_2) of $\mathfrak{I}^i_{(\cdot)}$-progressively measurable consumption and investment strategies is admissible for agent i if $(c^i,\pi^i_1,\pi^i_2) \in \mathscr{L}_+[0,T]\times\mathscr{L}^2[0,T]\times\mathscr{L}^2[0,T]$ and satisfies the non-negativity constraint $X^i_t \geq 0, \ t \in [0,T]$ where X^i solves the stochastic differential equation,

$$dX^i_t = [r_tX^i_t-c^i_t]dt + X^i_t\pi^i_{1t}[(\alpha^i_t-r_t)dt+\rho_{1t}dv^i_{1t}+\rho_{2t}dv^i_{2t}] + X^i_t\pi^i_{2t}[-r_tdt+\rho^Z_tdv^i_{2t}] + X^i_t[\gamma(\lambda_t)+\pi^i_{1t}\lambda_{1t}+\pi^i_{2t}\lambda_{2t}]dt, \tag{A.4}$$

with $X^i_0 = x^iS_0, \ t \in [0,T]$. No other constraints are placed on investment strategies of agents in this economy. A triplet (c^i,π^i_1,π^i_2) of admissible processes is optimal for preferences represented by U if there is no other admissible pair $(c^{i'},\pi^{i'}_1,\pi^{i'}_2)$ such that $U(c^{i'}) > U(c^i)$. Standard results on optimization in unconstrained markets (Karatzas, Lehoczky and Shreve (1987), Cox and Huang (1989, 1991)) enable us to compute these optimal demands stated below. The proof of Theorem A.1 follows from Theorem A.2 and Corollary A.3 in Detemple and Murthy (1994).

Theorem A.1: Consider the artificial economy with private investment opportunity sets (A.1)-(A.3) and without portfolio constraints. Define $\theta^i_{1t}(\lambda_t) \equiv (\rho_{1t})^{-1}[(\alpha^i_t+\lambda_{1t}-r_t)-\rho_{2t}(\rho^Z_t)^{-1}(\lambda_{2t}-r_t)]$, $\theta^i_{2t}(\lambda_t) \equiv (\rho^Z_t)^{-1}(\lambda_{2t}-r_t)$ and $\theta^i_t(\lambda_t) \equiv (\theta^i_{1t}(\lambda_t),\theta^i_{2t}(\lambda_t))$. Suppose that the condition (A.5) holds for i = 1,2, $t \in [0,T]$:

$$\lambda \in \mathbf{D}, \ r \in \mathscr{L}^1[0,T], \ \theta^i(\lambda) \in (\mathscr{L}^2[0,T])^2, \ E^i[\exp(\tfrac{1}{2}\int_0^t[(\theta^i_{1s}(\lambda_s))^2+(\theta^i_{2s}(\lambda_s))^2]ds)] < \infty. \tag{A.5}$$

Then, the optimal demands of agent i = 1,2 are given below with X^i solving (A.4):

$$c^i_t(\lambda) = mpc_tX^i_t, \tag{A.6}$$

$$\pi^i_{1t}(\lambda) = (\rho_{1t})^{-2}[(\alpha^i_t+\lambda_{1t}-r_t)-\rho_{2t}(\rho^Z_t)^{-1}(\lambda_{2t}-r_t)], \tag{A.7}$$

$$\pi_{2t}^i(\lambda) = (\rho_{1t}\rho_t^Z)^{-2}[(\lambda_{2t}-r_t)(\rho_t)^2-(\alpha_t^i+\lambda_{1t}-r_t)\rho_{2t}\rho_t^Z]. \tag{A.8}$$

Note that the solution to the optimization problem of section 1, (c^{i^*},π^{i^*}), if it exists, is admissible for agent i in the artificial economy. This follows since $\pi^{i^*} \in K$, and $\gamma(\lambda) \equiv \sup_{\pi\in K}(-\pi\lambda)$ implies $\gamma(\lambda_t)+\pi_{1t}^i\lambda_{1t}+\pi_{2t}^i\lambda_{2t} \geq 0$ for all admissible policies $(\pi_1^i,\pi_2^i) \in K$. By the optimality of (A.6)-(A.8), (c^{i^*},π^{i^*}) is also dominated in the artificial economy: $U(c^i(\lambda)) \geq U(c^{i^*})$ for all $\lambda \in D$.

Suppose now that there exists $\lambda^i \equiv (\lambda_1^i,\lambda_2^i) \in D$ such that (i) $(\pi_1^i(\lambda^i),\pi_2^i(\lambda^i)) \in K$, and (ii) $\gamma(\lambda_t^i)+\pi_{1t}^i(\lambda^i)\lambda_{1t}^i+\pi_{2t}^i(\lambda^i)\lambda_{2t}^i = 0$, $t\in[0,T]$. Note that $\gamma(\lambda^i) = \xi_S\text{Max}(\lambda_1^i,0) + (1+\xi_B)\text{Max}(-\lambda_1^i,0)$ for $\lambda^i \in D$. Using (A.7)-(A.8) it can be verified that conditions (i) and (ii) are satisfied if and only if $\lambda_t^i \equiv (\lambda_{1t}^i,\lambda_{2t}^i)$ is given by

$$\lambda_{1t}^i = \begin{cases} r_t-\alpha_t^i+(\rho_t)^2(1+\xi_B) \\ r_t-\alpha_t^i-(\rho_t)^2\xi_S \\ 0 \end{cases} \quad \text{and} \quad \lambda_{2t}^i = \begin{cases} r_t+(1+\xi_B)\rho_{2t}\rho_t^Z & \text{if } (\rho_t)^{-2}(\alpha_t^i-r_t) > 1+\xi_B \\ r_t-\xi_S\rho_{2t}\rho_t^Z & \text{if } (\rho_t)^{-2}(\alpha_t^i-r_t) < -\xi_S \\ r_t+(\rho_t)^{-2}(\alpha_t^i-r_t)\rho_{2t}\rho_t^Z & \text{otherwise.} \end{cases} \tag{A.9}$$

Then the policies $(c^i(\lambda),\pi^i(\lambda))$ of (A.6)-(A.8) evaluated at λ^i in (A.9) generate a wealth process (A.4) that is indistinguishable from the wealth process these same policies would generate in the constrained economy. Also since $\pi^i(\lambda^i) \in K$ it satisfies the borrowing and short sales constraints (10). Thus, $(c^i(\lambda^i),\pi^i(\lambda^i))$ is admissible in the constrained economy. By the optimality of c^{i^*} we have $U(c^{i^*}) \geq U(c^i(\lambda^i))$.

Combining these results we conclude that $c^{i^*} = c^i(\lambda^i)$ and that $\pi^{i^*} = \pi_i^i(\lambda^i)$. Thus, substituting (A.9) in (A.6)-(A.8) yields the optimal demands for the constrained problem in (13)-(14). Note that the leverage (short sales) constraint is binding on agent i = 1,2 when $(\rho_t)^{-2}[\alpha_t^i-r_t] > 1+\xi_B (< -\xi_S)$. Evaluating $\theta_{jt}^i(\lambda^i)$, j = 1,2, at λ_t^i in (A.9) we have $(\theta_{jt}^i(\lambda^i))^2 \leq \text{max}[(1+\xi_B)^2,(\xi_S)^2](\rho_{jt})^2$; hence (12) ensures that (A.5) is met. $\square$

Proof of Theorem 2.2: Using (13), clearing in the spot market for the consumption good $\Sigma_i c_t^i = D_t$ gives $X_t \equiv X_t^1 + X_t^2 = (\text{mpc}_t)^{-1}D_t$, $t \in [0,T]$. Combining the market clearing conditions $\Sigma_i\pi^iX^i = S$ and $\Sigma_i(1-\pi^i)X^i = 0$ yields $X = S$; hence, $S_t = (\text{mpc}_t)^{-1}D_t$, and $\alpha_t^i = \hat{\mu}_t^i+\beta$, $\rho_{1t} = \sigma_t$, $\rho_{2t} = 0$, $t \in [0,T]$, i = 1,2. Evaluating (14) yields the aggregate demand for the riskless asset: $\Sigma_i(1-\pi^i)X_t^i = \Sigma_i X_t^i[1-\text{Max}\{-\xi_S,\text{Min}[\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t),1+\xi_B]\}]$. Clearing in this market $\Sigma_i(1-\pi^i)X^i = 0$ gives the four conditions below, for i,j = 1,2, i ≠ j,

$$\Sigma_i X_t^i[1-\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t)] = 0, \text{ when } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) \in [-\xi_S,1+\xi_B],$$

$$X_t^i(-\xi_B) + X_t^j[1-\sigma_t^{-2}(\hat{\mu}_t^j+\beta-r_t)] = 0, \text{ when } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B \text{ and } \sigma_t^{-2}(\hat{\mu}_t^j+\beta-r_t) \in [-\xi_S,1+\xi_B],$$

$$X_t^i(1+\xi_S) + X_t^j[1-\sigma_t^{-2}(\hat{\mu}_t^j+\beta-r_t)] = 0, \text{ when } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) < -\xi_S \text{ and } \sigma_t^{-2}(\hat{\mu}_t^j+\beta-r_t) \in [-\xi_S,1+\xi_B],$$

and $$X_t^i(-\xi_B) + X_t^j(1+\xi_S) = 0, \text{ when } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B \text{ and } \sigma_t^{-2}(\hat{\mu}_t^j+\beta-r_t) < -\xi_S,$$

which upon simplification yield the interest rate in (19) according to whether $(\Delta,\delta) \in L_0, L_{iB}, L_{iS}$, or $L_{iB,jS}$, using

Assumption 2.1. By Walras' law, the stock market clearing condition is then automatically satisfied. Using Ito's Lemma, $X_t = (mpc_t)^{-1}D_t$, r in (19), and the wealth process

$$dX_t^i = X_t^i Max\{-\xi_S, Min[\sigma_t^{-2}(\hat{\mu}_t+\beta-r_t), 1+\xi_B]\}[(\hat{\mu}_t+\beta-r_t)dt+\sigma_t dv_{it}^i] + X_t^i(r_t-mpc_t)dt,$$

the distribution of wealth $\delta \equiv X^1/X$ has the representation $d\delta \equiv \mu^\delta dt + \sigma^\delta dv_1^i$ below, $t \in [0,T]$, $\delta_0 = x^1 \in (0,1)$:

$$d\delta_t = \begin{cases} \delta_t(1-\delta_t)^2\Delta_t^2\sigma_t^2 dt + \delta_t(1-\delta_t)\Delta_t\sigma_t dv_{1t}^1 & \text{for } (\Delta_t,\delta_t) \in L_0 \\ \delta_t[\Delta_t-\delta_t(1-\delta_t)^{-1}\xi_B]\sigma_t^2\xi_B dt + \delta_t\sigma_t\xi_B dv_{1t}^1 & \text{for } (\Delta_t,\delta_t) \in L_{1B} \\ -\delta_t[\Delta_t+\delta_t(1-\delta_t)^{-1}(1+\xi_S)]\sigma_t^2(1+\xi_S)dt - \delta_t\sigma_t(1+\xi_S)dv_{1t}^1 & \text{for } (\Delta_t,\delta_t) \in L_{1S} \\ \delta_t^{-1}(1-\delta_t)^2\sigma_t^2\xi_B^2 dt - (1-\delta_t)\sigma_t\xi_B dv_{1t}^1 & \text{for } (\Delta_t,\delta_t) \in L_{2B} \\ \delta_t^{-1}(1-\delta_t)^2\sigma_t^2(1+\xi_S)^2 dt + (1-\delta_t)\sigma_t(1+\xi_S)dv_{1t}^1 & \text{for } (\Delta_t,\delta_t) \in L_{2S} \\ \lim_{s\uparrow t} d\delta_s & \text{for } (\Delta_t,\delta_t) \in L_{1B,2S}\cup L_{1S,2B}. \end{cases} \quad (A.10)$$

From (17), note that none of the (closures of the) sets L_{iB}, L_{jS}, and $L_{iB,jS}$ share a common boundary with (the closures of) L_{jB}, L_{iS}, and $L_{jB,iS}$, $i,j = 1,2$, $i \neq j$, unlike other pairs of the (closures of) L_0, L_{iB}, L_{iS}, and $L_{iB,jS}$. From Assumption 2.3(ii), $(\Delta_0,\delta_0) \in L_0$ and hence $\Delta_0(1-\delta_0) \leq \xi_B$. For $(\Delta,\delta) \in L_0$ we have $d(\Delta(1-\delta)) \equiv (\mu^{\Delta 1}-\mu^\delta-\sigma_1^\Delta\sigma^\delta)dt + (\sigma_1^\Delta-\sigma^\delta)dv_1^1 + \sigma_2^\Delta dv_2^1$, where $(\mu_t^{\Delta i},\sigma_{1t}^\Delta,\sigma_{2t}^\Delta) \neq (0,0,0)$ by Assumption 2.3(i). It follows that for $\xi_B < \infty$

$$P^i\{(\Delta_t,\delta_t) \in L_{1B}, (\Delta_u,\delta_u) \in L_0, 0\leq u<t, \text{ for some } t \in (0,T]\}$$

$$= P^i\{\Delta_t(1-\delta_t) > \xi_B, (\Delta_u,\delta_u) \in L_0, 0\leq u<t, \text{ for some } t \in (0,T]\} > 0, i = 1,2.$$

It may similarly be verified that (Δ,δ) can enter L_{2B} or L_{iS} from L_0 with positive probability, and that (Δ,δ) can enter L_0 or $L_{iB,jS}$ from L_{iB} or L_{jS} with positive probability, for $i,j = 1,2$, $i \neq j$, $\xi_S < \infty$. A passage from L_0 to $L_{1B,2S}$ ($L_{2B,1S}$) or vice-versa is a probability zero event since the common boundary $\delta = 1+\xi_S/(1+\xi_S+\xi_B)$ ($\delta = \xi_B/(1+\xi_S+\xi_B)$) is a point (see Figure 1). Hence, the constraints bind on both agents with positive probability. The existence of a solution δ to (A.10) follows since its coefficients μ^δ and σ^δ satisfy random Lipschitz conditions (Protter (1990), Chp.5, Theorem 6) for Δ finite. Hence, δ is an Ito process; in particular, it has continuous sample paths.

It remains to verify that (i) the demand functions and (ii) the value functions evaluated at the equilibrium stock price and interest rate are well defined. Evaluating $\theta_{1t}^i(\lambda_t^i)$ and $\theta_{2t}^i(\lambda_t^i)$ at prices (18)-(19) and using (17), it can be verified that $\theta_{1t}^i(\lambda_t^i) \leq \sigma_t^{-1}|\hat{\mu}_t^1-\hat{\mu}_t^2| + \sigma_t$ and $\theta_2^i(\lambda^i) = 0$, $i = 1,2$. Hence, assumption 2.2 combined with Corollary 5.13 in Karatzas and Shreve (1988) establish that $\eta_{\lambda t}^i \equiv \exp[-\int_0^t\theta_{1s}^i(\lambda_s^i)dv_{1s}^i - (1/2)\int_0^t(\theta_{1s}^i(\lambda_s^i))^2 ds]$ is a P^i-martingale at equilibrium; similarly for η_t^i in (20). Since $\mu \in \mathcal{L}^2[0,T]$ we have $\hat{\mu}^i \in \mathcal{L}^2[0,T]$, $i = 1,2$. Hence at equilibrium $r \in \mathcal{L}^2[0,T]$ and $\theta_{1t}^i(\lambda_t^i) \in \mathcal{L}^2[0,T]$ so that $\lambda^i \in D$. It follows that the demand functions and the value functions are well defined, and that Q is an EMM. $\square$

<u>Proof of Proposition 2.1</u>: Since (Δ,δ) has continuous sample paths the interest rate is a continuous process in the interior of the sets L_0, L_{iB}, L_{iS}, $i = 1,2$. Consider the common boundary of L_0 and L_{1B} where $\Delta_t(1-\delta_t) = \xi_B$ and hence $r_t = \beta+\hat{\mu}_t^1-\sigma_t^2(1+\xi_B)$. Observe that the latter is equal to the limit of $r \equiv \beta+\hat{\mu}^2-\sigma^2+\xi_B\delta(1-\delta)^{-1}\sigma^2$, as $\Delta(1-\delta)$ $\downarrow \xi_B$, from within L_{1B}. Thus, the sample path of the interest rate is continuous at the common boundary of L_0 and L_{1B}. Similarly it can be verified that r is continuous at the common boundaries of L_0 and each of L_{2B}, L_{1S}, and L_{2S}. Now consider events in which $(\Delta_t,\delta_t) \in L_{iB,jS}$, $(\Delta_{t-},\delta_{t-}) \in L_{iB}$ and $(\Delta_{t+},\delta_{t+}) \in L_{jS}$, $i,j = 1,2$, $i \neq j$. The LCRL restriction yields

$$r_t = r_{t-} = \beta+\hat{\mu}_{t-}^i-\sigma_{t-}^2+\xi_B(X_{t-}^j/X_{t-}^i)\sigma_{t-}^2 = \beta+\hat{\mu}_t^i-\sigma_t^2+\xi_B(X_t^j/X_t^i)\sigma_t^2,$$

where the third equality follows from the continuity of the processes $\hat{\mu}^i$, σ^2, and X^i, $i = 1,2$. Similarly,

$$r_{t+} = \beta+\hat{\mu}_{t+}^i-\sigma_{t+}^2-(1+\xi_S)(X_{t+}^j/X_{t+}^i)\sigma_{t+}^2 = \beta+\hat{\mu}_t^i-\sigma_t^2-(1+\xi_S)(X_t^j/X_t^i)\sigma_t^2.$$

Recalling that $X_t^j/X_t^i = (1+\xi_S)/\xi_B$ and $|\Delta_t| > 1+\xi_S+\xi_B$ when $(\Delta_t,\delta_t) \in L_{iB,jS}$, the difference $r_{t+} - r_t = \hat{\mu}_t^i-\hat{\mu}_t^i-\sigma_t^2(1+\xi_B+\xi_S) > 0$ is the size of the jump in the interest rate at time t. It also follows that there is a negative jump of size $\hat{\mu}_t^i-\hat{\mu}_t^i+\sigma_t^2(1+\xi_B+\xi_S)$ when (Δ,δ) crosses $L_{iB,jS}$ from the "reverse" direction at time t, i.e. in events in which $(\Delta_t,\delta_t) \in L_{iB,jS}$, $(\Delta_{t-},\delta_{t-}) \in L_{jS}$ and $(\Delta_{t+},\delta_{t+}) \in L_{iB}$. $\square$

<u>Proof of Proposition 3.1</u>: Assume there exists a $\mathcal{I}_{(\cdot)}^i$-progressively measurable process z^i such that $S_t \equiv E_t^i[\int_t^T(m_u^i/m_t^i)D_u du] + E_t^i[\int_t^T(z_u^i/m_t^i)du]$. Hence, $S_tm_t^i + \int_0^t m_u^i D_u du + \int_0^t z_u^i du = E_t^i[\int_0^T m_u^i D_u du] + E_t^i[\int_0^T z_u^i du]$, $t \in [0,T]$, is a $(P^i,\mathcal{I}_{(\cdot)}^i)$-martingale; the gains process $\int_t^T[d(S_um_u^i) + m_u^i D_u du]$ has expected value $- E_t^i\int_t^T z_u^i du$. Using the cum-dividend stock returns in (18), the IMRS in (23), and Ito's Lemma, yields $z_t^i = -S_tm_t^i[\hat{\mu}_t^i+\beta-r_t-\gamma(\lambda_{1t}^i)-\sigma_t\theta_{1t}^i(\lambda_{1t}^i)]$; the definition of $\theta_{1t}^i(\lambda_{1t}^i)$ then implies $z_t^i = [\lambda_{1t}^i+\gamma(\lambda_{1t}^i)]S_tm_t^i$. Using the definition of $\gamma(\lambda_{1t}^i)$ leads to the decomposition $z_t^i = Max(\lambda_{1t}^i,0)S_tm_t^i + [\xi_S Max(\lambda_{1t}^i,0) + \xi_B Max(-\lambda_{1t}^i,0)]S_tm_t^i$. Thus, (24) follows in terms of the definitions (25)-(26). The interpretations of χ^i and κ^i as speculative gains and collateral services from the stock are discussed in the text following Proposition 3.1. It is obvious from (22) that λ_{1t}^i is positive (negative) in events in which the constraint on short sales of the stock (leverage) binds on agent i. $\square$

<u>Proof of Proposition 4.1</u>: Each agent's optimization problem in section 4, at the given prices (18)-(19) and (32), may be solved using the methodology of Cvitanic and Karatzas (1992) as in the proof of Theorem 2.1. These optimal policies are given by $c_t^i = mpc_tX_t^i$ where X_t^i satisfies (33) with optimal portfolio

$$\pi_{St}^i = \begin{cases} -\xi_S & \text{if } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) < -\xi_S \\ max\{-\xi_S, [\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t)-\phi_{Gt}(1+\xi_B)]/(1-\phi_{Gt})\} & \text{if } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B \end{cases} \tag{A.11}$$

29

$$\pi_{Bt}^i = \begin{cases} \max\{-\xi_B,\ 1+\xi_S-[\xi_S+\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t)]/\phi_{Gt}\} & \text{if } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) < -\xi_S \\ -\xi_B & \text{if } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B \end{cases} \tag{A.12}$$

$$\pi_{St}^i + (1-\pi_{St}^i-\pi_{Bt}^i)\phi_{Gt} = \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) \qquad \text{if } \sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) \in [-\xi_S, 1+\xi_B] \tag{A.13}$$

provided $\phi_{Gt} \neq 0$ when $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) < -\xi_S$ and $\phi_{Gt} \neq 1$ when $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B$. If $\phi_{Gt} = 0$ when $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t)$ $< -\xi_S$ ($\phi_{Gt} = 1$ when $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) > 1+\xi_B$) the optimal portfolio is described by $\pi_{St}^i = -\xi_S$ and $\pi_{Bt}^i \geq -\xi_B$ ($\pi_{St}^i \geq -\xi_S$ and $\pi_{Bt}^i = -\xi_B$). To show that the equilibrium of Theorem 2.2 is not valid it suffices to show that the optimal policies above do not clear markets in *some* events. Using (34), assume w.o.l.o.g that there exists some event (ω,t) in which $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) < -\xi_S$, $\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t) \in [-\xi_S, 1+\xi_B]$, and $\phi_{Gt} \neq 0$. Assume further, also w.o.l.o.g, that market clearing has been satisfied at all $u \in [0,t)$. Then, agent i's wealth $X^i(\omega,t)$, $i=1,2$, is the same in this event as that in the equilibrium of Theorem 2.2 and, since the interest rate is given by (19), satisfies

$$r_t = \beta+\hat{\mu}_t^2-\sigma_t^2(1+\xi_S)(X_t^1/X_t^2). \tag{A.14}$$

Next, note that in this event stock market clearing, $\pi_{St}^1 X_t^1 + \pi_{St}^2 X_t^2 = X_t$, requires

$$\pi_{St}^2 X_t^2 = X_t + \xi_S X_t^1, \tag{A.15}$$

while market clearing in the riskless asset, $\pi_{Bt}^1 X_t^1 + \pi_{Bt}^2 X_t^2 = 0$, requires

$$\pi_{Bt}^2 X_t^2 = -\max\{-\xi_B,\ 1+\xi_S-[\xi_S+\sigma_t^{-2}(\hat{\mu}_t^i+\beta-r_t)]/\phi_{Gt}\}X_t^1, \tag{A.16}$$

using (A.11)-(A.12) for agent 1's optimal portfolio. It may now be verified that agent 2's demands in (A.15)-(A.16) that are constructed to clear markets, are not optimal for him given the interest rate in (A.14): i.e. they do not satisfy (A.13). $\qquad\square$

Proof of Proposition 5.1: The solution (c^{i*}, π^{i*}) is characterized using the results of Cvitanic and Karatzas (1992). The optimal portfolio π^{i*} satisfies $\pi^{i*}\lambda_S^i + \gamma(\lambda^i) = 0$. Define $\theta^i(\lambda_t^i) \equiv \rho_t^{-1}(\alpha_t^i+\lambda_t^i-r_t)$ and the $\mathfrak{I}_{()}^W$-martingale $\eta^i(\lambda_t^i) \equiv \exp[-\int_0^t \theta^i(\lambda_s^i)dW_s - (1/2)\int_0^t(\theta^i(\lambda_s^i))^2 ds]$; let $J^i(\lambda_t^i) \equiv \exp\{-\int_0^t(r_s+\gamma(\lambda_s^i))ds\}\eta^i(\lambda_t^i)$. The optimal consumption policy $c_t^{i*} = H^i(y^i e^{\beta t}J^i(\lambda_t^i))$, where $H^i(.)$ is the inverse of $u_c^i(.)$ and y^i is a constant. This gives the IMRS of agent i: $m_{tu}^i \equiv m_u^i/m_t^i$ where $m_t^i \equiv e^{-\beta t}u_c^i(c_t^i)/u_c^i(c_0^i) = J^i(\lambda_t^i)$. Using the market clearing condition $\Sigma_i c^i = \Sigma_n D_n$ and Ito's Lemma we get (42)-(43). The first order conditions to the problem

$$\pi^{i*} \in \text{argmax } -[\pi^i\lambda_S^i + \gamma(\lambda^i)] \text{ subject to } \pi_n^i \geq -\xi_n^i,\ n=1,...,N, \text{ and } 1-\Sigma_n\pi_n^i \geq -\xi_B.$$

yield the characterization for λ_n^i, $n,m = 1,...,N$, $n \neq m$, in (38). The decomposition (39) follows as in the proof of Proposition 3.1. It may be verified that $\lambda_n^i + \gamma(\lambda^i) = \chi_n^i + \kappa_n^i$. The interpretations of χ^i and κ^i as speculative gains and collateral services from the stock are discussed in the text following Proposition 5.1. $\qquad\square$

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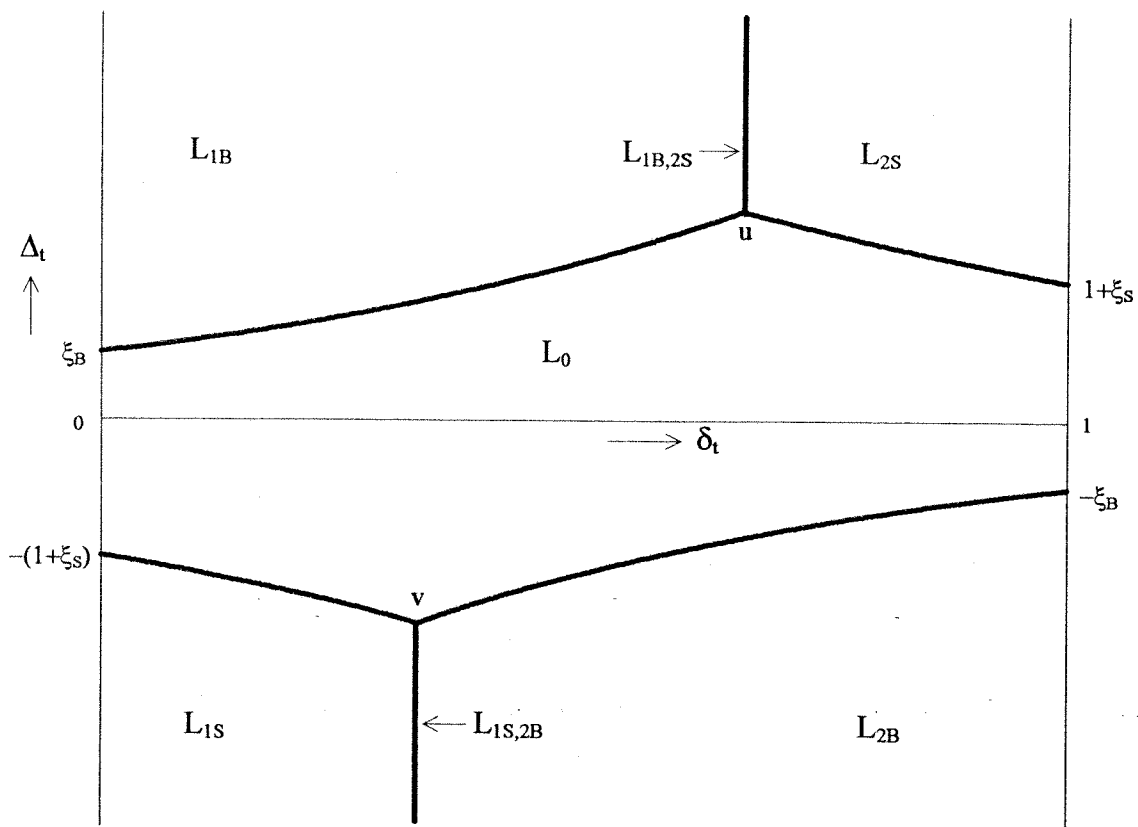
Figure 1: Possible Equilibrium Configurations

For $i,j = 1,2$, $i \neq j$: when $(\Delta_t, \delta_t) \in L_0$ both agents are unconstrained; when $(\Delta_t, \delta_t) \in L_{iB}$ agent i is borrowing constrained with agent j unconstrained; when $(\Delta_t, \delta_t) \in L_{iS}$ agent i is short sales constrained with agent j unconstrained; and when $(\Delta_t, \delta_t) \in L_{iB,jS}$ and $(\Delta_t, \delta_t) \in L_{iB}$ (L_{jS}) agent i is borrowing constrained with agent j unconstrained (agent j is short sales constrained with agent i unconstrained). The points "u" and "v" have the respective coordinates $((1+\xi_S)/(1+\xi_S+\xi_B), 1+\xi_S+\xi_B)$ and $(\xi_B/(1+\xi_S+\xi_B), -(1+\xi_S+\xi_B))$.
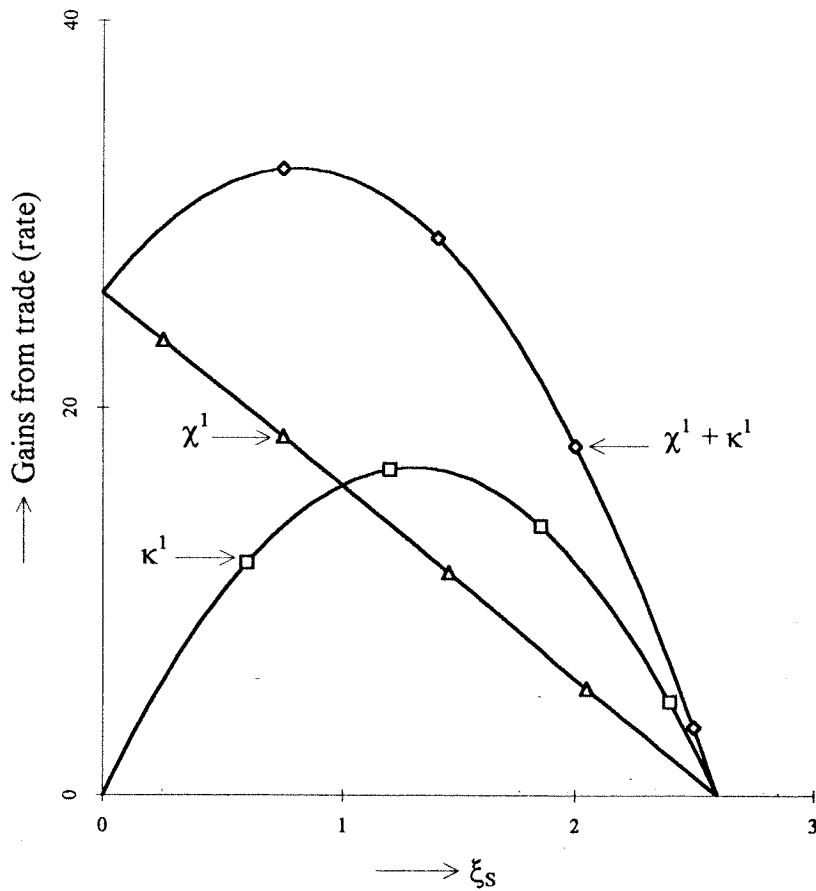
Figure 2: Increasing-decreasing expected gains from trade

The effects of relaxing the margin requirement on short sales of stock (i.e. increasing ξ_S) in event (ω,t) on the speculative gains $\chi^1(\omega,t)$, collateral services $\kappa^1(\omega,t)$, and expected gains from trade $\chi^1(\omega,t) + \kappa^1(\omega,t)$ that agent 1 imputes to the stock. Here, for small enough ξ_S, the marginal increase in collateral services on relaxation of the constraint more than offsets the marginal decrease in speculative gains; $\Delta(\omega,t) = -4$, $\delta(\omega,t) = 0.1$, $\sigma(\omega,t) = 0.3$, $S(\omega,t) = 100$.
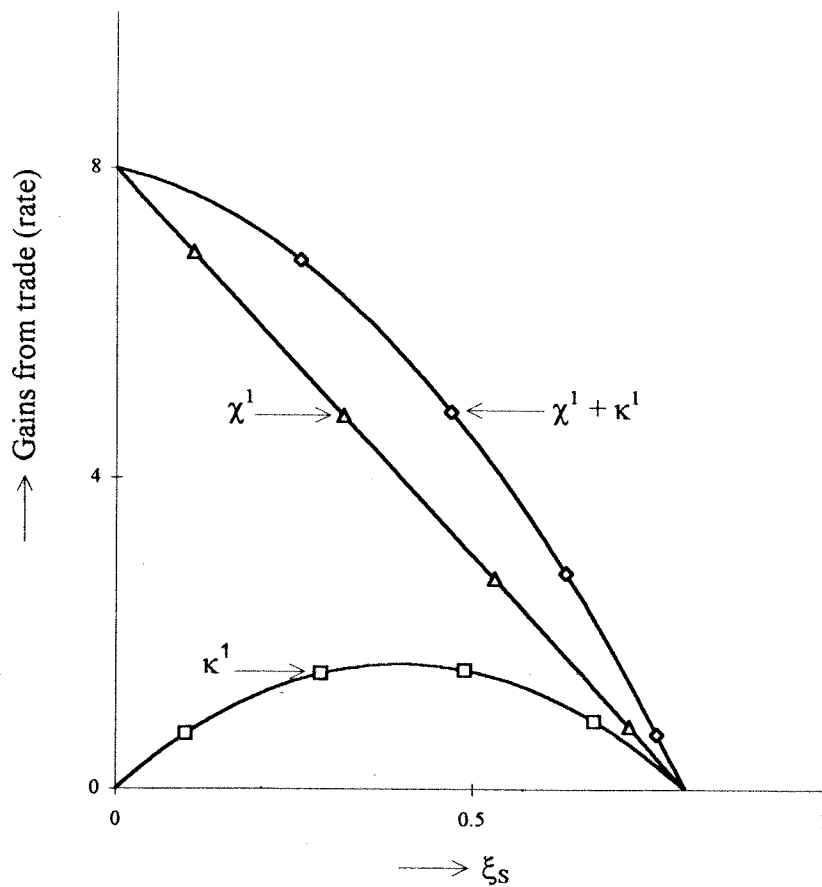
Figure 3: Decreasing expected gains from trade

The effects of relaxing the margin requirement on short sales of stock (i.e. increasing ξ_S) in event (ω,t) on the speculative gains $\chi^1(\omega,t)$, collateral services $\kappa^1(\omega,t)$, and expected gains from trade $\chi^1(\omega,t) + \kappa^1(\omega,t)$ that agent 1 imputes to the stock. Here, for all ξ_S, the marginal increase in collateral services on relaxation of the constraint never exceeds the marginal decrease in speculative gains; $\Delta(\omega,t) = -2$, $\delta(\omega,t) = 0.25$, $\sigma(\omega,t) = 0.3$, $S(\omega,t) = 100$.

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