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Kernel Autocorrelogram for Time Deformed Processes*

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Résumé / Abstract

L'objectif de cet article est de proposer une procédure d'estimation des autocorrélations pour les processus échantillonnés à des intervalles inégaux, modélisés comme processus subordonnés en temps continu. Ces processus, que l'on appelle aussi processus avec déformation du temps, ont été proposés dans plusieurs contextes. Avant d'élaborer sur la possibilité de modélisation des séries temporelles de ce type, on s'intéresse tout d'abord au diagnostic et à l'analyse des statistiques descriptives. Dans le domaine des processus en temps continu, cette difficile tâche peut être accomplie en ayant recours à la méthode d'estimation de l'autocorrélation par noyau. Cet article présente le cadre conceptuel, la procédure d'estimation et ses propriétés asymptotiques. Pour illustrer, un exemple empirique est aussi inclus.

The purpose of the paper is to propose an autocorrelogram estimation procedure for irregularly spaced data which are modelled as subordinated continuous time series processes. Such processes, also called time deformed stochastic processes, have been discussed in a variety of contexts. Before entertaining the possibility of modelling such time series one is interested in examining simple diagnostics and data summaries. With continuous time processes this is a challenging task which can be accomplished via kernel estimation. This paper develops the conceptual framework, the estimation procedure and its asymptotic properties. An illustrative empirical example is also provided.

Mots Clés : Processus subordonnés, Observations manquantes, Processus en temps continu, Méthodes non paramétriques

Keywords : Subordinated Processes, Irregularly Spaced Data, Continuous Time Processes, Nonparametric Methods

JEL : C14, C51

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1 Introduction

While the bulk of time series are recorded at regularly spaced intervals there are many cases where observations are irregularly spaced. Sometimes data are irregularly spaced because some observations are missing, in other cases it is the data generating mechanism which results in observations sampled at unequally spaced time intervals. We will be concerned with the latter class of models. Examples can be found in neurophysiological data (see e.g. Brillinger (1984)), observations collected from clinical experiments (see e.g. Jones (1984)), in physics (see Jones and Tryon (1983)) and economics, particularly applications involving financial markets data (see e.g. Clark (1973)).

We will focus on situations where a subordinated process framework is adopted. This framework is particularly attractive when the rate at which observations arrive is driven by some other process which is either latent or observable. This situation is often encountered with intra-daily financial markets applications where trading volume is a directing process for asset price movements. Indeed, the concept of subordinated stochastic processes, originated by Bochner (1960), was used by Mandelbrot and Taylor (1967) and later refined by Clark (1973) to explain the behavior of speculative prices. They argued that since the number of transactions in any particular period is random, one may think of asset price movements as the realization of a process $Y_t = Y^*_{Z_t}$ where Z_t is a directing process. This positive nondecreasing stochastic process Z_t can for instance be thought as related to either the number of transactions or to the cumulated trading volume, or to the cumulated volatility, or more fundamentally to the arrival of information.

Many applications of interest involve continuous time processes sampled at discrete dates, hence $Y^* : z \in \mathbb{R}^+ \to Y^*_z \in \mathbb{R}$ and the directing process $Z : t \in \mathbb{N} \to Z_t \in \mathbb{R}^+$. Typically they also involve very large data sets. For financial markets applications it can be thousands of data points for a single trading day. Before entertaining the possibility of modelling such series we want to compute autocorrelograms and other diagnostics. The large data sets make the case of nonparametric methods *feasible*; the context of continuous time processes with partial observability make nonparametric methods *necessary*. The purpose of our paper is to introduce kernel autocorrelogram estimation for irregularly spaced data modelled as discretely sampled continuous time subordinated processes with *observable* directing processes. Hence, we explore the autocorrelation properties of a process in continuous time which is driven by an observable Z_t. Section 2 presents a brief review of subordinated processes and their properties. Section 3 discusses the autocorrelogram estimation, while section 4 concludes with an application.

2 Properties of Subordinated Processes

In this section we will present certain properties of subordinated processes. Definitions and notations are fixed in section 2.1. We first consider in section 2.2 second order properties of the processes, namely: second order stationarity and the relation between the autocovariance functions of Y and Y^*. Next we study in section 2.3 some distributional properties such as strong stationarity and examine when a subordinated process is Markovian. We conclude with the example of a time deformed Ornstein-Uhlenbeck process in section 2.4.

2.1 Definitions and notations

We will make a distinction between calendar time denoted by t, and intrinsic time, the latter sometimes also called operational time, and denoted by z. We introduce the following notations:

i) the time changing process, called the directing process by Clark (1973), associates the operational scale with the calendar time. It is a positive strictly increasing process:

$$Z : t \in \mathbb{N} \longrightarrow Z_t \in \mathbb{R}^+; \tag{1}$$

ii) the process of interest evolving in the operational time is denoted by:

$$Y^* : z \in \mathbb{R}^+ \longrightarrow Y^*_z \in \mathbb{R}; \tag{2}$$

iii) finally we may deduce the process in calendar time $t \in \mathbb{N}$ by considering:

$$Y_t = Y^* \circ Z_t = Y^*_{Z_t}. \tag{3}$$

The introduction of a time scaling process is only interesting if the probabilistic properties of the process of interest become simpler in intrinsic time. It explains the introduction of the assumption below which ensures that all the links between the two processes (Y_t) and (Z_t) in calendar time result from the time deformation.

Assumption A.2.1: The two processes Z and Y^* are independent.

This assumption can be relaxed under certain circumstances, but it requires parametric model specifications which go far beyond the scope of autocorrelogram analysis considered here (see for instance Stock (1988), Ghysels and Jasiak (1994) for further discussion).

2.2 Second order properties

As usual for time series analysis we will first study the second order properties of the processes Y and Y^*. Assuming that both processes are second order integrable, we consider the first order moments:

$$\begin{cases} m(t) = E(Y_t), \ defined\ on\ \mathbb{N}, \\ m^*(z) = E(Y_z^*), \ defined\ on\ \mathbb{R}^+, \end{cases} \tag{4}$$

and the autocovariance functions:

$$\begin{cases} \gamma(t,h) = E[(Y_t - EY_t)(Y_{t+h} - EY_{t+h})] \ , \ t \in \mathbb{N}, h \in \mathbb{N}, \\ \\ \gamma^*(z_0, z) = E\left[\left[Y_{z_0}^* - E\left(Y_{z_0}^*\right)\right]\left[Y_{z+z_0}^* - E\left(Y_{z+z_0}^*\right)\right]\right] \ , \ z_0 \in \mathbb{R}^+, z \in \mathbb{R}^+. \end{cases} \tag{5}$$

From the definition of a time deformed process and by using the independence assumption between the two processes Z and Y^*, we can establish the following result (stated without proof):

Property 2.2.1: Under Assumption A.2.1:

$$\begin{aligned} m(t) &= E[m^*(Z_t)], \\ Cov(Y_t, Z_{t+h}) &= Cov(m^*(Z_t), Z_{t+h}), \\ \gamma(t,h) &= E[\gamma^*(Z_t, Z_{t+h} - Z_t)] + Cov[m^*(Z_t), m^*(Z_{t+h})]. \end{aligned}$$

It is possible now to discuss some sufficient conditions for the second order stationarity of the process Y. These conditions are *moment conditions* on the underlying process Y^*, and *distributional conditions* on the directing process Z.

Property 2.2.2: Under the independence Assumption A.2.1., the Y process in calendar time is second order stationary if the following assumptions are satisfied:

Assumption A.2.2: Y^* is second order stationary:

$$m^*(z) = m^*,\ \forall z,\ \gamma^*(z_0, z) = \gamma^*(z),\ \forall z_0, z.$$

Assumption A.2.3: The directing process has strongly stationary increments: the distribution of $\Delta_h Z_t = Z_{t+h} - Z_t$ is independent of t, $\forall h, t$.

This property can also be shown straightforwardly. A consequence of Property 2.2.2 is that we can have second order stationarity of the processes Y and Y^* simultaneously. In such a case we get: $m(t) = m^*, \gamma(t,h) = E[\gamma^*(\Delta_h Z_t)], Cov(Y_t,\ Z_{t+h}) = 0,\ \forall h$, and in particular Y and Y^* have the same marginal mean and variance, and we observe no correlation between the series Y and Z, while Y is a (stochastic) function of Z.

2.3 Strong stationarity and Markov properties

While it is natural to consider first the second order properties, it is obviously also of interest to study distributional properties of the two processes Y and Y^*, like strong stationarity or Markov properties. In this section we present only the main results; the proofs are given in Appendix 1.

For identification purpose we introduce the following normalization of calendar and intrinsic time clocks (see also Stock (1988)):

Assumption A.2.4: For $t = 0$, the intrinsic time is set at zero, i.e. $Z_0 = 0$.

Property 2.3.1 Let us assume Assumptions A.2.1 and A.2.4 hold. Then the process in calendar time is strongly stationary under the two following conditions:

Assumption A.2.5: Y^* is strongly stationary.

Assumption A.2.6: The directing process has strongly stationary multivariate increments, i.e. the distribution of $(\Delta_{t_1} Z_t, \ldots, \Delta_{t_n} Z_t)$ *is independent of* t for any $t, t_1, \ldots, t_n$.

Proof: see Appendix 1.

To examine the Markov properties of subordinated processes in calendar time we consider processes whose finite dimensional distributions have a pdf with respect to some dominating measures, chosen for convenience as the Lebesgue measure. Then assuming that Y^* and Z are each Markov processes of order one, the conditional pdf of Y_{t_n} and Z_{t_n} is given by:
$f[Y_{t_n} = y_n, Z_{t_n} = z_n | Y_{t_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}, ..., Y_{t_1} = y_1, Z_{t_1} = z_1] =$
$f[Y^*_{z_n} = y_n, Z_{t_n} = z_n | Y^*_{z_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}, ..., Y^*_{z_1} = y_1, Z_{t_1} = z_1] =$
$f[Y^*_{z_n} = y_n | Y^*_{z_{n-1}} = y_{n-1}, ..., Y^*_{z_1} = y_1] f[Z_{t_n} = z_n | Z_{t_{n-1}} = z_{n-1}, ..., Z_{t_1} = z_1]$,
where the latter follows from Assumption A.2.1. Then using the Markovian properties we obtain:

$$f[Y^*_{z_n} = y_n | Y^*_{z_{n-1}} = y_{n-1}] f[Z_{t_n} = z_n | Z_{t_{n-1}} = z_{n-1}].$$

Therefore the conditional distribution depends on the past values through the most recent ones $Y_{t_{n-1}}, Z_{t_{n-1}}$. Hence, we showed that:

Property 2.3.2: Under Assumption A.2.1, if Y^* *and* Z *are Markov processes of order one, then the joint process* (Y, Z) *is also a Markov process of order one.*

It is well known that, while the joint process (Y, Z) is Markovian, it does not necessarily imply that the marginal process Y is also Markovian of order one. However, this property is satisfied under the following additional conditions.

Property 2.3.3: Let the conditions of Property 2.3.2 hold. The process Y *is a Markov process of order one under the additional assumptions:*

Assumption A.2.7: The conditional distribution of $Y^*_{z+z_0}$ *given* $Y^*_{z_0} = y_0$ *only depends on* (z, z_0) *through* z.

Assumption A.2.8: The directing process has independent increments.

Proof: see Appendix 1.

Assumption A.2.7 means that the conditional pdf:
$f\left(Y^*_{z_n} = y_n \middle| Y^*_{z_{n-1}} = y_{n-1}\right) = f^*_n\left(y_n, y_{n-1}; z_n - z_{n-1}\right)$, i.e. depends on z_n and z_{n-1} through the difference $z_n - z_{n-1}$.

A byproduct of the proof is an expression of the transition kernel for the process Y, namely:

$$f(Y_{t_n} = y_n | Y_{t_{n-1}} = y_{n-1}) = E f^*_n(y_n, y_{n-1}; Z_{t_n} - Z_{t_{n-1}}),$$

and as usual we can easily check in this case that:
$f[Y_{t_n} = y_n | Y_{t_{n-1}} = y_{n-1}] = f[Y_{t_n} = y_n | Y_{t_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}]$, i.e. that (Z_t) does not Granger cause (Y_t) as discussed for instance in Florens and Mouchart (1985).

2.4 Time deformed Ornstein-Uhlenbeck processes

The Ornstein-Uhlenbeck process is of course the simplest example of a stationary continuous time process satisfying a diffusion equation. It will therefore be ideal for illustrating the properties we discussed in the previous sections. Moreover, it is worth noting that this type of process appears in continuous time finance applications particularly in stochastic volatility models. Ghysels and Jasiak (1994) for instance used a subordinated Ornstein-Uhlenbeck process to analyze a stochastic volatility model with a time deformed evolution of the volatility process. We will first examine the autocovariance structure of a subordinated Ornstein-Uhlenbeck process and show how time deformation affects the time dependence of the process. Typically, in intrinsic time these processes are the analogues of AR(1) processes whereas in discrete calendar time such processes have an ARMA representation with uncorrelated, though nonlinearly dependent, innovations.

The process Y^* is defined as the stationary solution of the stochastic differential equation:

$$dY^*_z = k\left(m - Y^*_z\right) dz + \sigma dW^*_z, \;\; k > 0, \sigma > 0, \tag{6}$$

where W^* is a standard Brownian motion indexed by $\mathbb{R}^+$, independent of the directing process. It is well known that Y^* is a Markov process of order one, and that the conditional distribution of $Y^*_{z+z_0}$ given $Y^*_{z_0}$ has a Gaussian distribution, with conditional mean:

$$E\left(Y^*_{z+z_0} \mid Y^*_{z_0}\right) = m + \rho^z \left(Y^*_{z_0} - m\right), \tag{7}$$

and conditional variance:

$$V\left(Y^*_{z+z_0} \mid Y^*_{z_0}\right) = \sigma^2 \frac{1-\rho^{2z}}{1-\rho^2}, \tag{8}$$

where: $\rho = \exp -k$. Using the independence Assumption A.2.1, we can rewrite (8) in calendar time as:

$$Y_t = m + \rho^{\Delta Z_t}\left(Y_{t-1} - m\right) + \left\{\sigma^2 \frac{1-\rho^{2\Delta Z_t}}{1-\rho^2}\right\}^{\frac{1}{2}} \epsilon_t, \tag{9}$$

where $\epsilon_t \sim I.I.N\,(0,1)$ and are independent of Z, and where $\Delta Z_t = Z_t - Z_{t-1}$. Moreover, we also have a similar relation for lag h:

$$Y_t = m + \rho^{\Delta_h Z_t}(Y_{t-h} - m) + \left\{\sigma^2 \frac{1-\rho^{2\Delta_h Z_t}}{1-\rho^2}\right\}^{1/2} \varepsilon_{h,t},$$

where $\varepsilon_{h,t} \sim N(0,1)$ and is independent of Z, and where $\Delta_h Z_t = Z_t - Z_{t-h}$. The previous relation gives the conditional distribution of Y_t given Y_{t-h} and the current and past values of the directing process. The conditional distribution of Y_t given Y_{t-h} is deduced by integrating out the directing process. To perform such an integration let us consider a directing process with iid increments. The conditions of Properties 2.3.1 and 2.3.3 are satisfied, so that the process in calendar time is strongly stationary and Markov of order one. Its autocovariance function is given by:

$$\gamma(h) = E\gamma^*(\Delta_h Z_t) = \gamma^*(0)\ E\left(\rho^{\Delta_h Z_t}\right) = \gamma^*(0)\ \left(E(\rho^{\Delta Z_t})\right)^h,$$

since (Z_t) is with iid increments. Hence, the process in calendar time has a weak AR(1) representation with an autoregressive coefficient $E(\rho^{\Delta Z_t})$ which is smaller than one. The conditional pdf is:

$$\begin{aligned} f(Y_t = y_t | Y_{t-1} = y_{t-1}) &= E f^*(y_t, y_{t-1}, \Delta Z_t) \\ &= \textstyle\int f^*(y_t, y_{t-1}; \Delta Z_t = \Delta z) g\left(\Delta z\right) d(\Delta z), \end{aligned}$$

where g is the pdf of ΔZ_t.

Since the heterogeneity introduced by the time deformation both affects the conditional mean and conditional variance, we immediately deduce that the conditional distribution of Y_t given Y_{t-1} is not Gaussian. Moreover, the value of, say the autoregressive coefficient depends on the distribution of ΔZ_t. To illustrate this, let us consider increments corresponding to a gamma process, i.e. $(Z_t) \sim \gamma(\nu t, \lambda)$; hence the increments ΔZ_t are mutually independent with identical distribution $\gamma(\nu, \lambda)$. It follows that:

$$r(\lambda, \nu, \rho) = E\left(\rho^{\Delta Z_t}\right) = \int_0^\infty (,\ (\nu))^{-1} \exp(-\lambda\mu)(\lambda\mu)^{\nu-1} \lambda\rho^\mu d\mu$$

$$= \int_0^\infty (,\ (\nu))^{-1} \exp(-(\lambda - \log\rho)\mu)(\lambda\mu)^{\nu-1} \lambda d\mu = (\lambda/(\lambda - \log\rho))^\nu.$$

It should be noted that $0 < \lambda/(\lambda - \log\rho) < 1$. Moreover, one can examine how $r(\lambda, \nu, \rho)$ varies with respect to the parameters λ and ν, and relates to ρ. Intuitively, since $EZ_t = \nu/\lambda$, we expect that deformed time on average accelerates or slows down relative to calendar time depending on whether ν/λ is larger or smaller than one. In addition, since $r(\lambda, \nu, \rho) = (1 + (\log\rho/(\lambda - \log\rho)))^\nu$, we note that it is an increasing function of λ, given ρ and ν, with $r \to 0$ as $\lambda \to 0$ and $r \to 1$ as $\lambda \to \infty$. As Figure 2.1 illustrates, we can consider a monotone and increasing mapping from λ to r. Hence, there is a unique $\lambda^*(\nu, \rho)$ such that $r[\lambda^*(\nu, \rho), \nu, \rho] = \rho$. If $\lambda < \lambda^*(\nu, \rho)$, then $r < \rho$, so that calendar time autocorrelation is weaker than deformed time autocorrelations. Moreover, it may be noted that $\lambda^*(\nu, \rho)$ is an increasing function of ν.

3 Kernel correlogram estimation

The literature on kernel nonparametric estimation of time series models is rather scant and focuses exclusively on equally spaced data generated by discrete time models (see e.g. Collomb (1981), Robinson (1983, 1988), Bierens (1983, 1987), Georgiev (1984), Altman (1990), Györfi et al. (1990), Härdle (1990), Härdle and Vieu (1992), Gouriéroux et al. (1994) among others). In this section we propose a kernel-based estimator of the intrinsic time autocorrelogram for continuous time processes suspected to have a subordinated representation driven by an observable Z_t. A first subsection is devoted to the definition of the intrinsic time autocorrelogram. A second subsection deals with its asymptotic distribution. A final subsection deals with confidence bounds.

3.1 Estimation of the intrinsic time correlogram

In section 2.2 we discussed the second order properties of subordinated stochastic processes, and we examined the links between autocovariance functions for Y and Y^*. In this section we assume that observations of Y_t and Z_t, $t = 1, ..., T$ are available and propose estimators for $\gamma(h)$ and $\gamma^*(z)$ under the assumption that Property 2.2.2 holds. Let us first recall that the empirical autocovariance function for a zero mean calendar time process can be written as:

$$\hat{\gamma}_T(h) \quad = \frac{1}{T}\sum_{t=1}^{T} Y_t Y_{t+h} = \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \mathbf{1}_{t-\tau=h} \approx \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \mathbf{1}_{t-\tau=h}}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} \mathbf{1}_{t-\tau=h}} \quad .$$

This formulation of the empirical autocovariance function would suggest an estimator for $\gamma^*(z)$ by analogy. The difficulty is that the autocovariance function γ^* is defined on the real line, whereas we only have a finite number of observations Z_t, $t = 1, ..., T$, therefore a small number of pairs (Z_t, Z_τ) such that $Z_t - Z_\tau = z$ given (this number is generally speaking equal to zero). This forces us to rely on smoothing through a kernel namely:

$$\hat{\gamma}_T^*(z) = \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \frac{1}{h_T} K\left[\frac{Z_t - Z_\tau - z}{h_T}\right]}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} \frac{1}{h_T} K\left[\frac{Z_t - Z_\tau - z}{h_T}\right]}, \tag{10}$$

where h_T is a bandwidth, depending on the size of the sample, and K is a kernel function. It is worth noting that, contrary to $\hat{\gamma}_T(h)$, the computation of $\hat{\gamma}_T^*(z)$ requires double sums, which may imply a large number of terms in practice. This numerical drawback may be circumvented by choosing a kernel with bounded support.

3.2 Asymptotic properties

Under certain regularity conditions the kernel autocorrelogram estimator defined in (3.1) is consistent and has an asymptotic normal distribution. We present this distribution and discuss the regularity conditions it takes to obtain the result. The proofs appear in Appendix 3. We start with a short subsection treating the regularity conditions.

3.2.1 Regularity conditions

Generally speaking, one must impose conditions on (1) the processes, (2) the kernel function $K(\cdot)$ and (3) the bandwidth h_T to ensure consistency and asymptotic normality. The purpose of this section is to highlight the conditions specifically related to the time deformation features of the process, particularly focusing on those related to (1).

Typically one assumes in the standard context of equally spaced observations that time series processes are strictly stationary and satisfy some mixing conditions. For instance, Robinson (1983) shows for a kernel time series regression strong consistency and asymptotic normality using α-mixing conditions (for definitions see e.g. McLeish (1974), White (1984), Bierens (1994) among others). Uniform consistency was shown by Bierens (1983) under slightly different conditions, namely ν-stability in L^2 with respect to a φ-mixing base. For the relationships to other concepts of stochastic stability, see Potscher and Prucha (1991) and Bierens (1994).

Since the kernel autocorrelogram $\hat{\gamma}_T^*$ depends jointly on (Y_t, Z_t) or equivalently on $[(Y_z^*), (Z_t)]$, it is clear that we need some regularity conditions on the joint behavior of such processes. In particular, we expect to need some mixing conditions for (Y_z^*), some independence between $[(Y_z^*), (Z_t)]$, and some assumptions implying that the directing process, while stochastic, mimicks on average a deterministic time deformation. Moreover, if we are also interested in the calendar time autocorrelogram $\hat{\gamma}_T$, we also need some mixing condition for (Y_t). Let us summarize some of the regularity conditions which we have already encountered in section 2 and add some new ones.

Assumption A.3.1: The processes Y^* and Z are independent.
Assumption A.3.2: The time origins coincide, $Z_0 = 0$.
Assumption A.3.3: Y^* is strongly stationary, with zero mean.
Assumption A.3.4: (Y_t, Z_t) is $\alpha-$mixing, with $\alpha-$mixing coefficients:

$$\alpha_j = \sup_{A\epsilon\mathcal{F}_{-\infty}^{t}, B\epsilon\mathcal{F}_{t+j}^{+\infty}} |P(A \cap B) - P(A)P(B)|$$

such that : $\exists\theta > 2 : \sum_{j=J}^{\infty} \alpha_j = O\left(J^{-\theta}\right)$, as $J \to \infty$.

Assumption A.3.5: The directing process has iid increments.

Assumption A.3.6: The directing process satisfies $0 < \sum_{k=1}^{+\infty} f_k^c(z) < +\infty$, $\forall z$, where $f_k^c(z)$ is the pdf of $\Delta_k Z_t$, and because of the iid increments: $\sum_{k=1}^{+\infty} f_k^c(z) = \sum_{k=1}^{+\infty} f^k(z)$, where f^k is the k^{th} convoluate of the pdf of ΔZ_t.

Consider $\lim_{dz\to 0} \frac{1}{dz} \sum_{t=1}^{\infty} P[Z_t\epsilon(z, z+dz)] = \sum_{k=1}^{+\infty} f_k^c(z)$, which is the coverage density function of z by the directing process. The last assumption A.3.6 can easily be understood by noting that we may only expect consistency and asymptotic normality for $\hat{\gamma}_T^*(z)$, for all the values z, with equivalent rates, if such a coverage density exists. This condition is critical with regard to the tail behavior of the distribution f. For instance if ΔZ_t has a gamma distribution $\gamma(\nu; \lambda)$, we get:

$$\sum_{k=1}^{\infty} f_k^c(z) = \lambda \sum_{k=1}^{\infty} \frac{1}{\Gamma(\nu k)} \exp -\lambda z (\lambda z)^{\nu k - 1} < +\infty,$$

while if ΔZ_t has a Cauchy distribution, we get:

$$\sum_{k=1}^{\infty} f_k^c(z) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{k}{1 + k^2 z^2} = +\infty\,.$$

Under Assumption A.3.5, or more generally whenever ΔZ_thas some ergodic properties (see Appendix 2), mixing conditions on Y^* are strongly linked with mixing conditions on Y. To understand this link, we will consider the case where γ^* has an asymptotic geometric decline, namely: $\exists A > 0, 0 < \rho < 1 : |\gamma^*(z)| < A\rho^z$. Then we know that: $\gamma(h) = E(\gamma^*(\Delta_h Z_t))$, and:

$$|\gamma(h)| \leq E\,|\gamma^*(\Delta_h Z_t)|$$

$$\leq A\ E\left(\rho^{\Delta_h Z_t}\right) = A\left(E\rho^{\Delta Z_t}\right)^h,$$

and we deduce the geometric decline of $\gamma(h)$.

3.2.2 Consistency and asymptotic normality

It is possible to establish different consistency results. We will focus on a weak form of consistency, sufficient to ensure the use of the autocorrelograms. Stronger forms of consistency could be derived, like uniform a.s. consistency, but are beyond the scope of our paper. The following theorem is proven in Appendix 3:

Theorem: The nonparametric kernel autocorrelogram estimator defined in (3.1) is: (a) consistent: $\hat{\gamma}_T^*(z)$ *converges in probability to* $\gamma^*(z)$*, and (b) asymptotically normal with distribution:*

$$\sqrt{Th_T}\left[\hat{\gamma}_T^*(z) - \gamma^*(z)\right] \overset{d}{\to} N\left(0, Var\left(Y_{z_0}^* Y_{z_0+z}^*\right)\left[\int K^2(\nu)d\nu\right] / \left[2\sum_{k=1}^{\infty} f_k^c(z)\right]\right),$$

where $f_k^c(z)$ *is the pdf of* $\Delta_k Z_t$*, under the following regularity conditions:*

(i) Assumptions A.2.1 through A.2.7 in section 2

(ii) Assumptions A.3.1 through A.3.6 in section 3.2.1

(iii) Technical conditions appearing in sections A.3.1 - A.3.3 of Appendix 3.

Proof: See Appendix 3.

It is interesting to note that:

$$V\left(Y_{z_0}^* Y_{z+z_0}\right) = V\left[Y_{z_0}^* E\left(Y_{z+z_0}^* \left|Y_{z_0}^*\right.\right)\right] + E\left[Y_{z_0}^{*2} V\left(Y_{z+z_0}^* \left|Y_z^*\right.\right)\right],$$

and therefore the asymptotic precision of this estimator depends on the conditional first and second order moments of Y^*.

3.3 Confidence bounds for the kernel autocorrelogram

For the calendar time autocorrelogram it is standard to compute confidence intervals for the autocorrelations associated with the different lags. The usual approach is to calculate bounds lag by lag, and estimate them under the null hypothesis that the covariances are equal to zero for the lags larger than the one of interest. A similar approach can be followed for the intrinsic time autocorrelogram. The pointwise asymptotic result derived in the previous theorem is sufficient to derive the theoretical point wise bounds, equal to:

$$\pm 2\sigma = \pm \frac{2}{\sqrt{Th_T}} \frac{Var\left(Y^*_{z_o} Y^*_{z_o+z}\right)^{\frac{1}{2}} \left[\int K^2(v)\, dv\right]^{\frac{1}{2}}}{2\sum\limits_{k=1}^{\infty} f^c_k(z)}$$

For equally spaced calendar time autocorrelograms the confidence bounds only depend on the lag through the numerator. For irregularly spaced data it also depends on the denominator through the coverage density. Hence, confidence bounds do not necessarily decrease monotonely for large lags. To compute the confidence bounds we need a consistent estimator of σ, under a suitable null hypothesis of no correlation, which avoids an additional functional estimation of the density functions appearing in the denominator and of the cross moment $Var\left(Y^*_{z_o}, Y^*_{z_o+z}\right)$.

It is worth recalling that in the usual context of equally spaced data the autocorrelation coefficient $\hat{\rho}_T(h)$ can be recovered as the estimated regression coefficient $\hat{a}_T$ in the following regression:

$$Y_t = aY_{t-h} + u_t. \tag{11}$$

Moreover under the null hypothesis $H_{0,h} = \{\rho(h) = 0\} = \{a = 0\}$, we can obtain an estimate of the variance of $\hat{a}_T$ using a White (1980) correction. The White correction would be used for estimating the appropriate variance of $\hat{a}_T$, realizing that $Var\ \hat{a}_T$ is proportional to $(\sum\limits_{t=1}^{T} Y_{t-h})^{-2} \sum\limits_{t=1}^{T}\sum\limits_{\tau=1}^{T} \hat{u}_t \hat{u}_\tau Y_{t-h} Y_{\tau-h}$, where the $\hat{u}_t's$ are the OLS residuals. We proceed along the same lines in the case of continuous time subordinated processes. Namely, by analogy with the equally spaced case covered in (3.2), we can consider the quantity:

$$\hat{\rho}_T(z) = \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_\tau^2 K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)}.$$

Since asymptotically:

$$\frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_\tau^2 K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)} \simeq EY_t^2 = EY_z^{*2},$$

we directly note that:

$$\hat{\rho}_T(z) \sim \frac{\hat{\gamma}_T(z)}{\hat{\gamma}_T(0)}.$$

Moreover this estimator $\hat{\rho}_T(z)$ may be obtained as a Generalized Least Squares (GLS) estimator based on the following artificial regression:

$$Y_t = aY_\tau + u_{t,\tau}(z) \qquad \tau = 1, .., .T, t = 1, \ldots, T,$$

where $u_{t,z}(z)$ are zero-mean, uncorrelated and have variance: $V(u_{t,\tau}(z))$ $= \left[\frac{1}{h_T} K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right]^{-1}$. This artificial regression may also be written under a matrix form as:

$$\mathcal{Y}_T = a\mathcal{X}_T + u_T(z),$$

where $\mathcal{Y}_T = (\iota_T \otimes (Y_1, \cdots, Y_T))'$, $\mathcal{X}_T = ((Y_1, \cdots, Y_T) \otimes \iota_T)'$ and Var $u_T(z)$ =diag $\left(K\left((Z_t - Z_\tau - z)/h_T\right)^{-1}\right) = \Omega_T$.

The GLS estimator of a coincides with $\hat{\rho}_T(z)$. Its variance is given by

$$V\hat{a}_T(z) = (\mathcal{X}_T'\Omega_T^{-1}\mathcal{X}_T)^{-1}\mathcal{X}_T'\Omega_T^{-1}\Omega^*\Omega_T^{-1}\mathcal{X}_T(\mathcal{X}_T'\Omega_T^{-1}\mathcal{X}_T)^{-1},$$

where Ω^* is the true variance-covariance matrix of the error term. It is possible to avoid a direct estimation of this matrix Ω^* by using the White correction based on the residuals $\hat{u}_t(z) = Y_t - \hat{a}_T(z) Y$, and by replacing Ω^* by the matrix $\hat{\Omega}_T^*$ whose element are $\hat{u}_{t,\tau}(z)\,\hat{u}_{t',\tau'}(z)$. Therefore, a consistent estimator of the variance of $\hat{\rho}_T(z)$ is:

$$\hat{V}\hat{\rho}_T(z) = \left[\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_\tau^2 K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right]^{-2} \times$$

$$\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{t'=1}^{T}\sum_{\tau'=1}^{T} Y_\tau Y_{\tau'} \hat{u}_{t,\tau}(z)\,\hat{u}_{t',\tau'}(z)\, K\left(\frac{Z_t - Z_\tau - z}{h_T}\right) K\left(\frac{Z_{t'} - Z_{\tau'} - z}{h_T}\right)$$

Such an estimator is difficult to implement, however, because of the large number of terms appearing in the second sum. This difficulty may be circumvented if we follow the standard approach for autocorrelograms which consists of estimating the variance under the null hypothesis:

$$\tilde{H}_{o,z} = \{\rho(z') = 0; \forall z' \geq z\}.$$

Indeed, under $\tilde{H}_{o,z}$, we may consider the simplified estimator:

$$\hat{V}_o\hat{\rho}_T(z) = \left[\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_\tau^2 K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right]^{-2} \times$$

$$\sum_{t=1}^{T}\sum_{\tau=1}^{T}\sum_{t'=1}^{T}\sum_{\tau'=1}^{T} \left\{Y_\tau Y_{\tau'} Y_t Y_{t'} K\left(\frac{Z_t - Z_\tau - z}{h_T}\right) K\left(\frac{Z_{t'} - Z_{\tau'} - z}{h_T}\right) \mathbf{1}_{|Z_t - Z_{t'}| < z}\right\},$$

by using the fact that the constrained residual $\hat{u}_{t,\tau}(z)$ is equivalent to Y_t and the absence of correlation between Y_t and $Y_{t'}$ when $|Z_t - Z_{t'}|$ is larger than z.

4 Empirical examples

We consider applications involving financial markets data. The example involves daily return and trading volume data from the New York Stock Exchange (NYSE). Hence, we use equally spaced data in calendar time, where the cumulated volume series fulfills the role of the directing process.

The NYSE data covers the entire year 1992, a total of 232 trading days of observations of the daily returns of the S&P500 index and the daily volumes of trades on the NYSE. The data have been trended adjusted and non-trading day effects were also removed. The data transformations we undertook are standard and are described in Gallant, Rossi and Tauchen (1992). Figures 4.1 and 4.2 display plots of the return and volume series. The mean and variance for the return series are 0.014 and 0.2692 and for the volume series 9.3192 and 0.0119. As will be discussed in detail later we will use the daily average volume $m_V = 9.3192$ to compute the lags in the kernel autocorrelations. The kurtosis and skewness for the returns are 0.0955 and -0.0087 while the volume series exhibit more asymmetry (0.3018) and features thicker tails (1.0262).

The calculations involve two types of kernels. First, we consider a Gaussian kernel with a bandwidth of 0.0184. The bandwidth selection procedure was based on the Sheather and Jones (1991) method. It is obtained as the solution to the fixed point equation:

$$h = \left[R\left(K\right)/nR\left(\hat{f}''_{g(h)}\right)\left(\int uK\left(u\right)du\right)^{2}\right]^{1/5},$$

where $R\left(a\right) = \int au^2 du$ and $g\left(h\right)$ denotes the pilot bandwidth. This approach is known as "Solve the Equation" method (see e.g. Jones, Marron and Sheather (1994)). We also considered a second type of kernel, a straightforward one which greatly simplified the calculations. It amounts to computing:

$$\hat{\gamma}_T\left(z\right) = \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \mathbf{1}_{\left(\left|Z_t - Z_\tau - z\right| < h_T\right)}}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} \mathbf{1}_{\left(\left|Z_t - Z_\tau - z\right| < h_T\right)}}$$

Table 4.1: Autocorrelations and Kernel Autocorrelations
Daily Returns NYSE with Trading Volume Directing Process

Sample Jan-Dec. 1992

LAGS*	Standard Autocorrelations	Kernel Autocorrelations Gaussian	Kernel Autocorrelations Bounded
0	1.00	1.00000000	1.00000000
.25	-	.96109012	.98912166
.50	-	.85545780	.04118434
.75	-	.71047791	.02053376
1.00	-0.07	.55632985	-.05664217
1.25	-	.41449688	-.00990475
1.50	-	.29487185	-.02156523
1.75	-	.19983379	.04182950
2.00	-0.03	.12864804	-.02136905
2.25	-	.07835496	-.21960051
2.50	-	.04311944	-.31013754
2.75	-	.01566146	-.21134475
3.00	-0.04	-.00898645	-.19482959
4.00	-0.02	-.04963489	.16279907
5.00	-0.00	-.10870389	.22826619
6.00	-0.07	-.24866865	-.25052412
7.00	-0.01	-.37000344	.12951130
8.00	-0.08	-.50365109	-.55722679
9.00	-0.08	-.56670803	.36045349
10.00	0.00	-.92926302	-.51138681

Notes: The lags for the standard autocorrelations are based on daily observations. For the kernel autocorrelations lag x corresponds to $z = xm_V$, where m_V is the average daily trading volume.

Table 4.1 contains the autocorrelograms estimated from the data. The first column displays the standard autocorrelation function computed from the daily data. The next two columns show the autocorrelations for the returns subordinated to detrended trading volume series. A calendar time lag x corresponds to $z = xm_V$ where m_V is the average daily trading volume. Hence, in Table 4.1. we present the calendar time and time deformed autocorrelations side-by-side for integer values of $x = 1, ..., 10$, i.e. up to ten trading days. The trading volume distribution is not symmetric and has fat tails, as noted before and as shown more explicitly in Figure 4.3.

The results in Table 4.1 show remarkable differences between the calendar time and operational time autocorrelation functions. Indeed, the standard ACF shows that returns show very little temporal autocorrelation. The first order autocorrelation for instance is negative and small. In contrast the first order correlation of the subordinated representation is high regardless of the kernel being used. The insensitivity with regard to the kernel choice does unfortunately not carry through for the other lag specifications. With the Gaussian kernel we obtain a slowly declining ACF function. With the bounded support kernel it tapers off quite quickly but picks up again at a lag of $3m_V$.

Appendix 1

Proofs of Strong Stationarity and Markov Property

A.1.1 Strong stationarity

For any integrable function $g\left(Y_{t_1},...,Y_{t_n}\right)$, we get:

$$E\ g\left[Y_{t_1},...,Y_{t_n}\right]$$

$$= E\ E\left(g\left(Y_{t_1},...,Y_{t_n}\right)|Z_{t_1},...,Z_{t_n}\right)$$

$$= E\ E\left(g\left(Y^*_{Z_{t_1}},...,Y^*_{Z_{t_n}}\right)|Z_{t_1},...,Z_{t_n}\right)$$

$$= E\left(g\left(Y^*_0,Y^*_{Z_{t_2}-Z_{t_1}},...,Y^*_{Z_{t_n}-Z_{t_1}}\right)|Z_{t_1},...,Z_{t_n}\right);$$

the latter follows from the independence between Y^* and Z, and strong stationarity of Y^*. Furthermore, the above expression equals:

$$E\left(g\left(Y^*_0,Y^*_{Z_{t_2}-Z_{t_1}},...,Y^*_{Z_{t_n}-Z_{t_1}}\right)\right),$$

which, using Assumption A.6. yields:

$$E\left(g\left(Y^*_0,Y^*_{Z_{t_2-t_1}},...,Y^*_{Z_{t_n-t_1}}\right)\right) = E\ \left(g\left(Y_0,Y_{t_2-t_1},...,Y_{t_n-t_1}\right)\right).$$

Q.E.D.

A.1.2 Markov Property

Let us consider the conditional pdf:

$$f[Y_{t_n}=y_n|Y_{t_{n-1}}=y_{n-1},\ldots,Y_{t_1}=y_1] = \frac{f[Y^*_{Z_{t_n}}=y_n,Y^*_{Z_{t_{n-1}}}=y_{n-1},...,Y^*_{Z_{t_1}}=y_1]}{f[Y^*_{Z_{t_{n-1}}}=y_{n-1},...,Y^*_{Z_{t_1}}=y_1]}$$

$$= \frac{Ef[Y^*_{Z_{t_n}}=y_n,Y^*_{Z_{t_{n-1}}}=y_{n-1},...,Y^*_{Z_{t_1}}=y_1|Z_{t_1},...,Z_{t_n}]}{Ef[Y^*_{Z_{t_{n-1}}}=y_{n-1},...,Y^*_{Z_{t_1}}=y_1|Z_{t_1},...,Z_{t_{n-1}}]}$$

$$
\begin{aligned}
= \quad & E\{f[Y^*_{Z_{t_n}} = y_n | Y^*_{Z_{t_{n-1}}} = y_{n-1}, Z_{t_1}, \ldots, Z_{t_n}] \times \\
& f[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 | Z_{t_1}, \ldots, Z_{t_{n-1}}]\} / \\
& Ef[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 | Z_{t_1}, \ldots, Z_{t_{n-1}}]\} \\
= \quad & \frac{E\{f^*_n(y_n, y_{n-1}; Z_{t_n} - Z_{t_{n-1}}) f[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 | Z_{t_1}, \ldots, Z_{t_{n-1}}]\}}{Ef[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 | Z_{t_1}, \ldots, Z_{t_{n-1}}]} \\
& = Ef^*_n(y_n, y_{n-1}; Z_{t_n} - Z_{t_{n-1}}),
\end{aligned}
$$

where the latter equalities follow from Assumptions A.2.7 and A.2.8 respectively.

Q.E.D.

Appendix 2

Geometric declines of γ and γ^*

We discussed in subsection 3.2.1 that, when the increments are i.i.d., the geometric decline of γ^* implies the geometric decline of γ. We will verify in this appendix that this property holds under weaker assumptions. In particular, let us consider:

Assumption : The increments ΔZ_t of the directing process satisfy:

i) (ΔZ_t) *is strongly stationary;*

ii) $\left(, \sqrt{h} \right)^{-1} \sum_{k=1}^{h} \left(\Delta Z_{t+k} - \mu \right) \overset{d}{\to} N\left(0, 1\right),$

where: $\mu = E\left(\Delta Z_t\right)$, $,^2 = , (0) + 2 \sum_{k=1}^{\infty} , (k) < \infty$, with $, (k) =$ $cov\left(\Delta Z_t, \Delta Z_{t+k}\right).$

iii) There exists a positive constant c such that :
$P\left[\left(, \sqrt{h} \right)^{-1} \sum_{k=1}^{h} \left(\Delta Z_{t+k} - \mu \right) > c\sqrt{h} \right] \sim 1 - \Phi\left(c\sqrt{h} \right),$

when $h \to \infty$, where Φ is the c.d.f of the standard Normal distribution.

As in the proof of subsection 3.2.1, we know that:

$$
\begin{aligned}
\left|\gamma\left(h\right)\right| &\leq \; E\left|\gamma^*\left(\Delta_h Z_t\right)\right| \\
&\leq \; A \; E\left(\rho^{\Delta_h Z_t}\right) \\
&\leq \; A \; \left\{ E\left[\rho^{\Delta_h Z_t} \mathbf{1}_{\Delta_h Z_t > h(\mu + c\Gamma)}\right] + \; E\left[\rho^{\Delta_h Z_t} \mathbf{1}_{\Delta_h Z_t < h(\mu + c\Gamma)}\right] \right\} \\
&\leq \; A \; P\left[\Delta_h Z_t > h\left(\mu + c, \right)\right] + A \; \rho^{h(\mu + c\Gamma)} \\
&\leq \; A^* \left[1 - \Phi\left(c\sqrt{h} \right) \right] + A \; \rho^{(\mu + c\Gamma)h}
\end{aligned}
$$

$$\leq \; A^{**} \exp -\tfrac{c^2}{2} h + A \; \rho^{(\mu + c\Gamma)h},$$

which establishes the geometric decline of the γ function. Q.E.D.

It is worth noting that (omitting the technical aspects) we have approximately:

$$\begin{aligned} A \; E\left(\rho^{(\Delta_h Z_t)}\right) \;&\simeq\; AE \;\left(\rho^{\mu h + \Gamma\sqrt{h}u}\right), \text{ where } u \text{ is standard normal,} \\ &=\; AE \exp\left[\mu h \log\rho + , \; \log\rho\sqrt{h}u\right] \\ &=\; A \exp\left(\mu h \log\rho + \tfrac{\Gamma^2(\log\rho)^2}{2} h\right) \\ &=\; A\left(\exp\log\rho\left[\mu + \tfrac{\Gamma^2 \log\rho^2}{2}\right] h\right), \end{aligned}$$

which gives an idea of the new rate of geometric decline: $r = \rho^{\mu + \left(\Gamma^2 \log\rho\right)/2}$. It is larger or smaller than ρ depending on the sign of $\mu + \left(, {}^2 \log\rho\right)/2$, i.e. of the magnitude of the variance-covariance term $, {}^2$ in comparison with μ.

Appendix 3

Asymptotic properties of the kernel autocorrelogram estimator

We first determine the asymptotic first and second order moments and deduce the stochastic convergence of $\hat{\gamma}_T^*(z)$ to $\gamma^*(z)$. Next, we turn our attention to asymptotic normality of the estimator. A preliminary section is devoted to technical conditions, the next one covers the first and second order moments and finally a third one is devoted to the asymptotic distribution.

A.3.1.Technical conditions

In sections 2 and 3 we listed and discussed a number of regularity conditions. The purpose of this section is to complement these with a set of technical conditions. They are as follows:

(i) *Lipschitz conditions*

$(L1)\, |g_k(z+h_T\nu) - g_k(z)| \leq G_k(z)\, h_T\, |\nu|\,,$
with: $g_k(z) = \gamma^*(z)\, f_k^c(z)\,,$ and $\sum\limits_{k=-\infty}^{+\infty} G_k(z) < +\infty.$

$(L2)\, |\tilde{g}_k(z+h_T\nu) - \tilde{g}_k(z)| \leq \tilde{G}_k(z)\, h_T\, |\nu|\,,$ with: $\tilde{g}_k(z) = \,,^*(z)\, f_k^c(z)\,,$ and
$\sum\limits_{k=-\infty}^{+\infty} \tilde{G}_k(z) < +\infty$

(ii) *Kernel conditions*

$(K1)$ K is continuous with bounded support;
$(K2)$ $\int K(u)\, du = 1;$
$(K3)$ $\int uK(u)\, du = 0;$
$(K4)$ $\int |u|\, K(u)\, du < +\infty;$
$(K5)$ $\int |u|\, K^2(u)\, du < +\infty;$

(iii) *Bandwidth conditions*

$(B1)$ $h_T \to 0, \qquad$ as $T \to +\infty$;
$(B2)$ $Th_T \to +\infty$, as $T \to +\infty$;

A.3.2.Asymptotic first and second moments

To establish the consistency of the first and second order moments, let us write the estimator as:

$$\hat{\gamma}_T^*(z) = g_{1T}(z)/g_{2T}(z),$$

where:

$$g_{1T}(z) = T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau h_T^{-1} K\left[\frac{Z_t - Z_\tau - z}{h_T}\right],$$

$$g_{2T}(z) = T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{T} h_T^{-1} K\left[\frac{Z_t - Z_\tau - z}{h_T}\right].$$

We will focus primarily on $g_{1T}(z)$, since $g_{2T}(z)$ can be viewed as a special case of the former.

i) First order moments

$$E(g_{1T}(z)) = T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left[Y_t Y_\tau\, h_T^{-1} K\left[\frac{Z_t - Z_\tau - z}{h_T}\right]\right]$$

$$= T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left[Y_0^* Y_{Z_{t-\tau}}^*\, h_T^{-1} K\left[\frac{Z_{t-\tau} - z}{h_T}\right]\right],$$

because of the stationarity of the Y^* process and the stationarity of the increments of the Z_t process. Moreover, using Assumption A.3.4, we know that $Z_0 = 0$, and therefore $Y_{Z_0}^* = Y_0$. This yields:

$$E(g_{1T}(z)) = T^{-1}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left[\gamma^*(Z_{t-\tau})\, h_T^{-1} K\left[\frac{Z_{t-\tau} - z}{h_T}\right]\right]$$

$$= T^{-1}\sum_{k=-(T-1)}^{T-1} (T - |k|)\, E\left[\gamma^*(Z_k)\, h_T^{-1} K\left[\frac{Z_k - z}{h_T}\right]\right]$$

$$= 2T^{-1}\sum_{k=1}^{T-1}(T-|k|)\int \gamma^*(u)\ h_T^{-1}K\left[\frac{u-z}{h_T}\right]f_k^c(u)\,du,$$

using the property $\gamma^*(-z) = \gamma^*(z)$. As we take the limit $T\to\infty$, we expect that: $Eg_{1T}(z) \to \gamma^*(z)\,2\sum_{k=1}^{\infty} f_k^c(z)$.

To check this, let us examine the difference:

$$\lim_{T\to\infty}\left[E\left(g_{1T}(z) - \gamma^*(z)\,2\sum_{k=0}^{+\infty} f_k^c(z)\right)\right]$$

$$= \lim_{T\to\infty}\{\ 2T^{-1}\sum_{k=1}^{T-1}(T-|k|)\int[\gamma^*(z+h_T\nu)\,f_k^c(z+h_T\nu)$$

$$-\gamma^*(z)\,f_k^c(z)]K(\nu)\,d\nu\ \}$$

$$+\lim_{T\to\infty}\left\{\gamma^*(z)\,2\left[\sum_{k=1}^{T-1} f_k^c(z) - T^{-1}\sum_{k=1}^{T-1}(T-|k|)\,f_k^c(z)\right]\right\}.$$

From the regularity conditions A.3.6 on the density functions $f_k^c(.)$, we know that the second limit on the right hand side is zero.

Indeed we get:

$$\sum_{k=1}^{T-1}\frac{|k|}{T}f_k^c(z)$$
$$= \sum_{k=1}^{\sqrt{T}}\frac{|k|}{T}f_k^c(z) + \sum_{k=\sqrt{T}+1}^{T-1}\frac{|k|}{T}f_k^c(z)$$
$$\le \frac{1}{\sqrt{T}}\sum_{k=1}^{\sqrt{T}} f_k^c(z) + \sum_{k>\sqrt{T}} f_k^c(z)$$
$$\le \frac{1}{\sqrt{T}}\sum_{k=1}^{\infty} f_k^c(z) + \sum_{k>\sqrt{T}} f_k^c(z),$$

and this last quantity tends to zero because of A.3.6.

To show that the first limit on the right hand side also equals zero, we rely on the Lipschitz condition $(L1)$ listed in section A.3.1. It yields:

$$\left|T^{-1}\sum_{k=-(T-1)}^{T-1}(T-|k|)\int\left[\gamma^*(z+h_T\nu)\,f_k^c(z+h_T\nu) - \gamma^*(z)\,f_k^c(z)\right]K(\nu)\,d\nu\right|$$

$$\leq T^{-1} \sum_{k=-(T-1)}^{T-1} (T - |k|)\, G_k\,(z)\, h_T \left(\int |\nu|\, K\,(\nu)\, d\nu \right).$$

At $T \to \infty$ the right hand side converges to zero (using the summability of $G_k(z)$ and condition (K4) on the kernel). To summarize, so far we have established that:

$$\lim_{T\to\infty} E\,(g_{1T}\,(z)) = \gamma^*\,(z)\, 2 \sum_{k=1}^{+\infty} f_k^c\,(z)\,.$$

Similarly, we get:

$$\lim_{T\to\infty} E\,(g_{2T}\,(z)) = 2 \sum_{k=1}^{+\infty} f_k^c\,(z)\,.$$

ii) Second order moments

We now turn our attention to the second order properties. We focus again exclusively on $g_{1T}\,(z)$ and study its normalized variance. However, instead of tackling immediately the normalized variance we will first focus on $E\,[\sqrt{Th_T}(g_{1T}\,(z) - \; E\,(g_{1T}\,(z)\,|\sigma_T(z)))]^2$ with $\sigma_T(z)$ the $\sigma-$ algebra generated by $(Z_t)_{t=1}^{T}$, and show it has a finite limit as $T \to \infty$. The latter expression can be written as:

$$E\left[\sqrt{Th_T}\,(g_{1T}\,(z) - E\,(g_{1T}\,(z)\,|\sigma_T\,(z)\,))\right]^2 = C1 + C2,$$

where:

$$C1 = E\, \left\{ (Th_T)^{-1} \sum_{t=1}^{T} \sum_{\tau=1}^{T} ,^{*}\,(Z_{t-\tau})\, K^2 \left[\frac{Z_{t-\tau} - z}{h_T} \right] \right\},$$

$$C2 = E\, \left\{ (Th_T)^{-1} \sum_{t=1}^{T} \sum_{t'\neq t}^{T} \sum_{\tau=1}^{T} \sum_{\tau'\neq\tau}^{T} \left(Y^*_{Z_t} Y^*_{Z_\tau} - \gamma^*\,(Z_t - Z_\tau)\right)\right.$$

$$\left.\left(Y^*_{Z_{t'}} Y^*_{Z_{\tau'}} - \gamma^*\,(Z_{t'} - Z_{\tau'})\right) K \left[\frac{Z_t - Z_\tau - z}{h_T} \right] K \left[\frac{Z_{t'} - Z_{\tau'} - z}{h_T} \right] \right\},$$

where: $, ^*(z) = E\ [Y_z^* - \gamma^*(z)]^2$. We will show that $C1$ has a finite limit while $C2 \to 0$ as $T \to \infty$.

Let us first write $C1$ as:

$$C1 = T^{-1} \sum_{k=-(T-1)}^{T-1} (T - |k|)\, h_T^{-1}\ E\left(, ^*(Z_k)\, K^2\left[\frac{Z_k - z}{h_T}\right]\right)$$

$$= T^{-1} \sum_{k=-(T-1)}^{T-1} (T - |k|) \int , ^*(z + h_T\nu)\, f_k^c(z + h_T\nu)\, K^2(\nu)\, d\nu,$$

where by convention for negative k, Z_k denotes $-Z_{|k|}$, and f_k^c the associated density function.

Using Lipschitz condition $(L2)$ and kernel condition $(K5)$ we can easily establish that $C1$ has a finite limit. To show that $C2 \to 0$ as $T \to \infty$ let us rewrite $C2$ as:

$$C2 \;=\; (Th_T)^{-1}\ E \sum_{t=1}^{T} \sum_{t' \neq t}^{T} \sum_{\tau=1}^{T} \sum_{\tau' \neq \tau}^{T} E\ [, ^*(Z_{\tau-t}, Z_{t'-t}, Z_{\tau'-t})] \times K\left[\frac{Z_{t-\tau} - z}{h_T}\right] K\left[\frac{Z_{t'-\tau'} - z}{h_T}\right],$$

where $, ^*(.,.,.)$ is defined in analogy to $, ^*(\cdot)$. The above quadruple sum can be reduced to a triple sum, namely:

$$C2 = (Th_T)^{-1} \sum_{k=-(T-1)}^{T-1} \sum_{k'=-(T-1)}^{T-1} \sum_{k''=-(T-1)}^{T-1} (T - |k|)\, E\ [\Gamma^*(Z_k, Z_{k'}, Z_{k''}) \times$$

$$K\left(\frac{Z_k - z}{h_T}\right) K\left(\frac{Z_{k'} - Z_{k''} - z}{h_T}\right)$$

$$= (Th_T)^{-1} \sum_{k=-(T-1)}^{T-1} \sum_{k'=-(T-1)}^{T-1} \sum_{k''=-(T-1)}^{T-1} (T - |k|)\, E$$

$$\int\int\int \Gamma^*(z + h_T\nu, z + h_T\nu', h_T\nu'')$$

$$f^c_{k,k',k''}\left(z+h_T\nu, z+h_T\nu', h_T\nu''\right) K\left(\nu,\nu',\nu''\right) d\nu d\nu' d\nu'',$$

with similar conventions as before for the negative values of k, k', k''. Using a Lipschitz condition argument similar to the previous one yields an expression involving h_T^3/h_T^2 which results in $C2 \to 0$ as $T \to \infty$.

So far we considered $\sqrt{Th_T}E\ \left[\left(g_{1T}(z) - E\ \left(g_{1T}(z)\right)|\sigma_T(z)\right)\right]^2$, and showed it had a finite limit as $T \to \infty$. Let us define $u_{1T}(z) = g_{1T}(z) - E\ \left(g_{1T}(z)|\sigma_T(z)\right)$. It is relatively easy to show that $E\ u_{1T}(z) = 0$ and that $E\ u_{1T}(z)^2 = (Th_T)^{-1} C$, where $C < +\infty$ so that $u_{1T}(z) \to 0$ in the L^2 norm. The above computations can also be applied to $g_{2T}(z)$. Together they yield:

$$V_{as}\left(\sqrt{Th_T}\left\{\begin{array}{c} g_{1T}(z) - \lim_T E\ g_{1T}(z) \\ g_{2T}(z) - \lim_T E\ g_{2T}(z) \end{array}\right\}\right)$$

$$= \lim_T \left\{\frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left(\left[\begin{array}{c} Y_tY_\tau \\ 1 \end{array}\right][Y_tY_\tau\ \ 1]\frac{1}{h_T}K^2\left[\frac{Z_t - Z_\tau - z}{h_T}\right]\right)\right.$$

$$-\frac{h_T}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left[\left[\begin{array}{c} Y_tY_\tau \\ 1 \end{array}\right]\frac{1}{h_T}K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right]$$

$$\left. E\left[\left[\begin{array}{c} Y_tY_\tau \\ 1 \end{array}\right]\frac{1}{h_T}K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right]\right\}$$

$$= \lim_T \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left(\left[\begin{array}{c} Y_tY_\tau \\ 1 \end{array}\right][Y_tY_\tau\ \ 1]\frac{1}{h_T}K^2\left[\frac{Z_t - Z_\tau - z}{h_T}\right]\right)$$

$$= \lim_T \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} E\left[\left(\begin{array}{cc} Y^{*2}_{Z_t}Y^{*2}_{Z_\tau} & Y^*_{Z_t}Y^*_{Z_\tau} \\ Y^*_{Z_t}Y^*_{Z_\tau} & 1 \end{array}\right)\frac{1}{h_T}K^2\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right],$$

which follows because the second term is asymptotically negligeable.

Let us introduce the quantity $\upsilon^*(z) = E\left(Y_{z_o}^{*2} Y_{z_o+z}^{*2}\right)$. We directly deduce after a change of variable similar to the one previously used that:

$$B(z) = \int K^2(\upsilon)\, d\upsilon\; 2 \sum_{k=1}^{\infty} f_k^c(z) \begin{bmatrix} \upsilon^*(z) & \gamma^*(z) \\ \gamma^*(z) & 1 \end{bmatrix}.$$

iii) Consistency in probability and asymptotic variance of $\hat{\gamma}_T^*(z)$.

The asymptotic properties derived on first and second order moments of $g_{1T}(z)$ and $g_{2T}(z)$ imply the stochastic convergence of these two sequences, and also the stochastic convergence of $\hat{\gamma}_T^*(z)$. The asymptotic variance of $\hat{\gamma}_T^*(z)$ is then deduced by the δ-method. We get:

$$V_{as}\left(\sqrt{Th_T}\left[\widehat{\gamma}_T^*(z) - \gamma(z)\right]\right) = \left[\upsilon^*(z) - \gamma^*(z)^2\right]$$
$$\left[\int K^2(\upsilon)\, d\upsilon\right] / \left(2 \sum_{k=1}^{\infty} f_k^c(z)\right),$$

where :

$$\upsilon^*(z) - \gamma^*(z)^2 = E\ \left(Y_{z_o}^{*2} Y_{z+z_o}^{*2}\right) - E\ \left(Y_{z_o}^* Y_{z+z_o}^*\right)^2 = V\left(Y_{z_o}^* Y_{z+z_o}^*\right).$$

A.3.3. Asymptotic normality

The asymptotic normality of the kernel autocorrelogram may be derived from the joint asymptotic normality of $\sqrt{Th_T}\left[g_{1T}(z) - Eg_{1T}(z), g_{2T}(z) - Eg_{2T}(z)\right]'$, by usual arguments.

i) A decomposition of $g_{1T}(z)$

Now let us examine the expression of a difference such as $\sqrt{Th_T}\left(g_{1T}(z) - Eg_{1T}(z)\right)$. We get:

$$\sqrt{Th_T}\left(g_{1T}(z) - Eg_{1T}(z)\right) =$$

$$\frac{1}{\sqrt{Th_T}} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \left[Y_{Z_t}^* Y_{Z_\tau}^* K\left(\frac{Z_t - Z_\tau - z}{h_T}\right) - E\left(Y_{Z_t}^* Y_{Z_\tau}^* K\left(\frac{Z_t - Z_\tau - z}{h_T}\right)\right)\right]$$

$$= \sum_{k=-(T-1)}^{T-1} \frac{1}{\sqrt{Th_T}} \sum_{t=Max(1,1-k)}^{Min(T,T-k)} \left(Y^*_{Z_t} Y^*_{Z_{t+k}} \; K\left(\frac{Z_t - Z_{t+k} - z}{h_T}\right) \right.$$
$$\left. - \, E\left(Y^*_{Z_t} Y^*_{Z_{t+k}} \; K\left(\frac{Z_t - Z_{t+k} - z}{h_T}\right)\right)\right)$$

$$= \sum_{k=-(T-1)}^{T-1} \frac{1}{\sqrt{Th_T}} \sum_{t=Max(1,1-k)}^{Min(T,T-k)} \left(Y^*_{Z_t} Y^*_{Z_t+\Delta_k Z_t} K\left(\frac{\Delta_k Z_t - z}{h_T}\right) \right.$$
$$\left. - \, E\left(Y^*_{Z_t} Y^*_{Z_t+\Delta_k Z_t} K\left(\frac{\Delta_k Z_t - z}{h_T}\right)\right)\right)$$

$$= \sum_{k=-(T-1)}^{T-1} \frac{\sqrt{Min(T,T-k) - Max(1,1-k)}}{\sqrt{T}} \quad U_{k,T},$$

where $U_{k,T}$ is simply a kernel regression of $\tilde{Y}_{kt} = Y^*_{Z_t} Y^*_{Z_t+\Delta_k Z_t}$ on $\tilde{Z}_{kt} = \Delta_k Z_t$. Therefore it will be possible to apply usual results concerning the asymptotic normality of such regressograms, as soon as the regressors $\tilde{Y}_k$ and the regressand $\tilde{Z}_k$ satisfy suitable regularity conditions.

ii) Asymptotic normality of the kernel regression estimator

We may for instance consider the set of regularity conditions introduced by Robinson (1983) [Theorem 5.3] for deriving the asymptotic normality of the kernel regression estimator of $\left(\tilde{Y}_t\right)$ on $\left(\tilde{Z}_t\right)$. We denote $g(z) = E\left(\tilde{Y}_t \mid \tilde{Z}_t = z\right)$.

(N1) $\left(\tilde{Y}_t, \tilde{Z}_t\right)$ is $\alpha-$mixing, with $\alpha-$mixing coefficients such that: $\exists \theta > 2 : \sum_{j \equiv J}^{\infty} \alpha_j = O\left(J^{-\theta}\right)$, as $J \to \infty$

(N2) $E\left|\tilde{Y}_t\right|^{\theta} < \infty$, with the same θ as in (N1)

(N3) The pdf of $\left(\tilde{Z}_t\right)$ is strictly positive:

(N4) The regression function is such that there exist:

$$\lambda > 0, z_o, c < +\infty, \; \textit{such that} \;\; :$$

$$|g(z_o - z) - g(z_o) - P_r(z)| < C\,|z|^{\lambda},$$

where r is the greatest integer less than λ and P_r is a polynomial in z of degree r.

(N5) The conditional second order moment $E\left(\tilde{Y}_t^2 \mid \tilde{Z}_t = \cdot\right)$ is continuous in z.

(N6) $E\left(\left|\tilde{Y}_t\right|^\gamma \mid Z_t = \cdot\right)$is locally bounded in a neighborhood of z for a $\gamma > \theta$.

iii) Asymptotic normality of $g_{1T}(z)$.

To derive the asymptotic normality of $g_{1T}(z)$, we have first to impose the set of assumption (N1)-(N6) to the various processes $\tilde{Y}_{kt} = Y_t Y_{t+k}$, $\tilde{Z}_{kt} = \Delta_k Z_t$. It may be noted that assumption (N1) written for the various process $\left(\tilde{Y}_{kt}, \tilde{Z}_k\right)$ $k = 1...$ is implied by assumption A.3.4 in section 3.2.1. The assumption (N2) implies in particular the existence of fourth order moments of Y_t. Assumption (N3) means that the various pdf $f_k^c(z), k = 1, ...$ are all strictly positive, and not only their sum (see assumption A.3.6).

Under these assumptions the asymptotic normality of $\sqrt{Th_T}\left[g_{1T}(z) - Eg_{1T}(z)\right]$ results from the asymptotic normality of the finite sums

$$\sum_{k=-K}^{K} \frac{\sqrt{\min(T,T-k) - \max(1,1-k)}}{\sqrt{T}} U_{k.T},$$

and from the fact that the remaining terms of the series may be bounded in probability by a term tending to zero with K.

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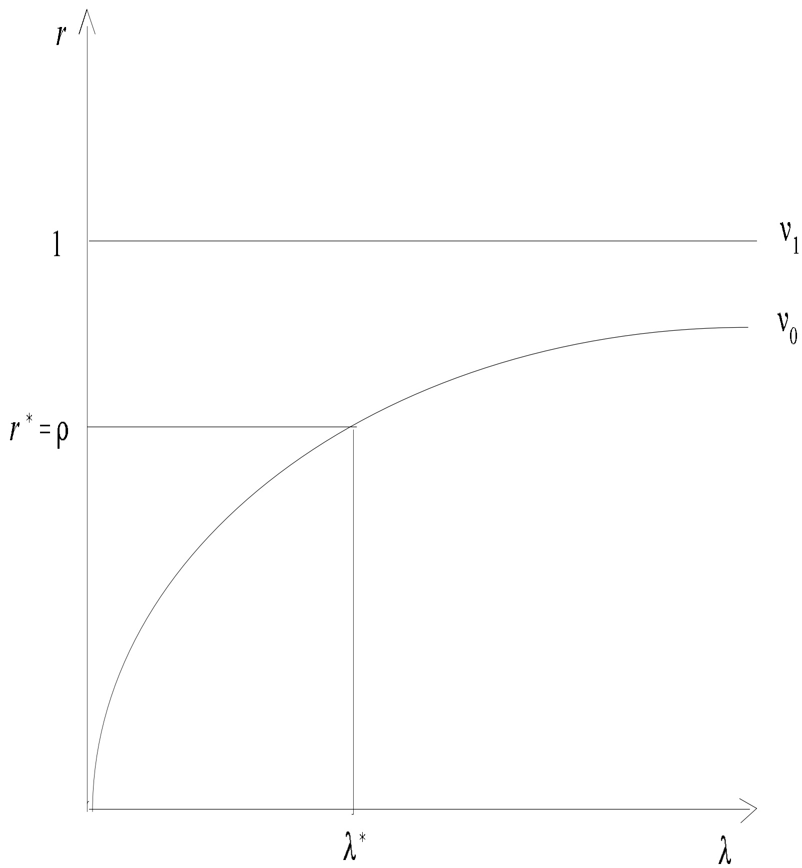


Figure 2.1: First order calendar time autocorrelation of time deformed Ornstein-Uhlenbeck process with gamma directing process.

Fig 4.1: S&P 500 (Adjusted)

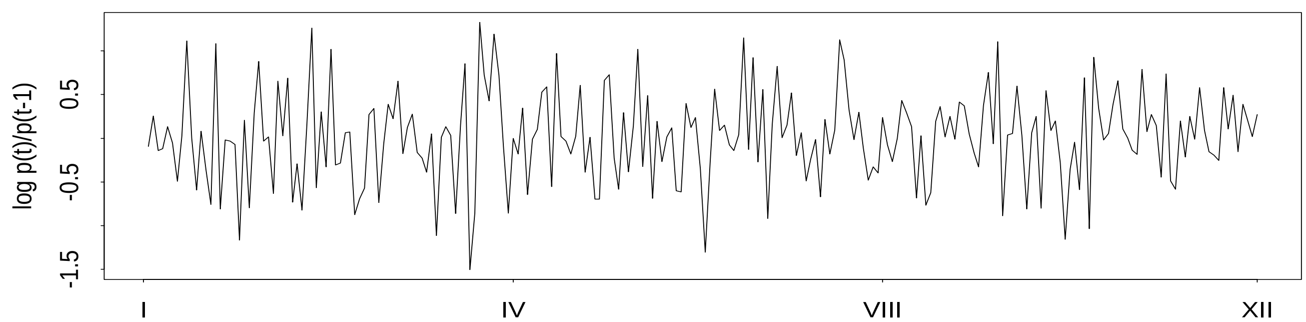


Fig 4.2: Volume NYSE (Adjusted)

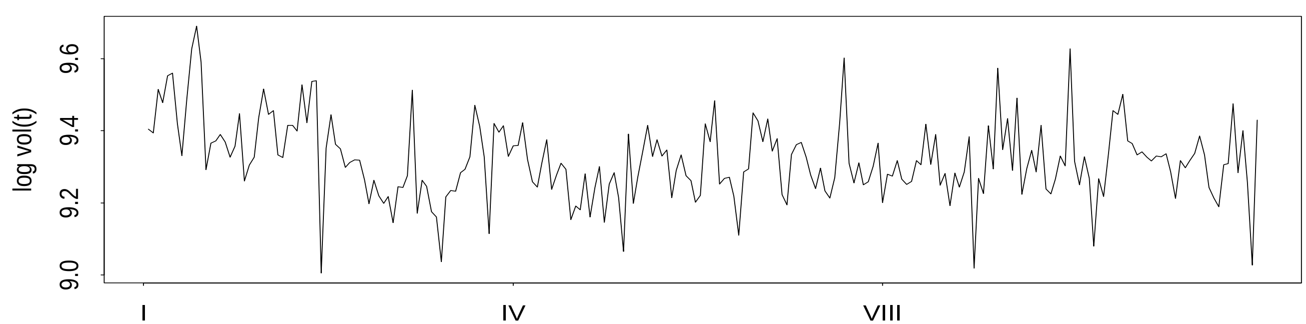
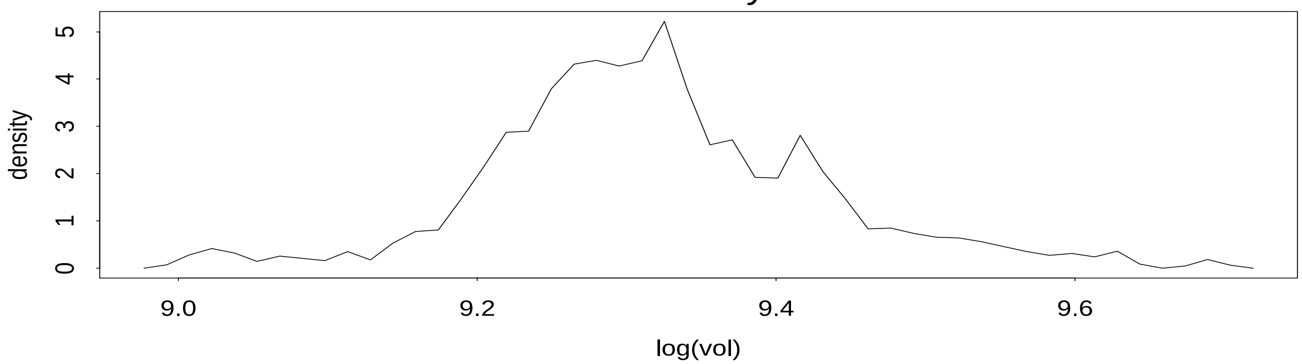


Fig 4.3: Volume NYSE (Adjusted)
density



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