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Jin-Chuan Duan, Jean-Guy Simonato

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Estimating and Testing Exponential-Affine Term Structure Models by Kalman Filter*

Jin-Chuan Duan†, *Jean-Guy Simonato*‡

Résumé / Abstract

Cette recherche propose une approche unificatrice pour l'estimation des paramètres de modèles de structure de taux d'intérêt de la classe exponentielle-affine. Cette famille de modèles, caractérisée par Duffie et Kan (1993), contient entre autres les modèles de Vasicek (1977), Cox, Ingersoll et Ross (1985) et Chen et Scott (1992). La méthode proposée utilise un filtre de Kalman approximatif qui requiert la spécification de l'espérance et de la variance conditionnelle du système. La méthode utilise simultanément plusieurs séries de rendements et permet l'ajout d'erreurs de mesure pour chaque serie. Une étude de simulation indique que la méthode proposée est fiable pour des échantillons de taille modérée. Une étude empirique utilisant trois modèles différents de la classe exponentielle-affine est présentée.

This paper proposes a unified state-space formulation for parameter estimation of exponential-affine term structure models. This class of models, charaterized by Duffie and Kan (1993), contains models such as Vasicek (1977), Cox, Ingersoll and Ross (1985) and Chen and Scott (1992), among others. The proposed method uses an approximate linear Kalman filter which only requires specifying the conditional mean and variance of the system in an approximate sense. The method allows for measurement errors in the observed yields to maturitiy, and can simultaneously deal with many yields on bonds with different maturities. A Monte Carlo study indicates thet the proposed method is a reliable procedure for moderate sample sizes. An empirical analysis of three existing exponential-affine term structure models is carried out using monthly U.S. Treasury yield data with four different maturities. Our test results indicate a strong rejection of all three models.

Mots Clé : Structure à Terme, Filtre de Kalman, Exponentielle-affine, Modèle State-Space, Quasi-maximum de vraisemblance, Test du Multiplicateur de Lagrange

Key Words : Term Structure, Kalman Filter, Exponential-Affine, State-Space Model, Quasi-Maximum Likelihood, Lagrange Multiplier Test

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† McGill University and CIRANO

‡ Université du Québec à Montréal and CIRANO

1. Introduction

The term structure of interest rates describes the relationship between the yield on a default–free debt security and its maturity. Given the high correlation among bond yields of different maturities, many theoretical models attempt to use a small number of factors to explain these joint movements. The typical macro–econometric approach is to specify a time–series model for the short–term interest rate and then employ the expectation hypothesis to derive a structural time series model for bond yields of different maturities. Examples of this approach, such as Hamilton (1987) and Hall, Anderson and Granger (1992), are abundant in the literature.

A different modelling approach, popular in the finance literature and pioneered by Vasicek (1977) and Dothan (1978), starts out by assuming a diffusion process for the instantaneous spot interest rate. Arbitrage arguments are then used to facilitate the derivation of a bond pricing formula. According to these models, the bond price is a function of the unobserved instantaneous spot interest rate and the model's parameters. A more general approach is to assume a set of unobserved state variables and proceed to derive the bond price as a function of these state variables by arbitrage and/or equilibrium arguments. Cox, Ingersoll and Ross (1985) (hereafter CIR), Richard (1978), Longstaff and Schwartz (1992), and Chen and Scott (1992) are some examples. Recently, Duffie and Kan (1993) have provided a characterization for the class of exponential–affine term structure models, which contains most of the aforementioned term structure models as special cases. Among the existing term structure models, Vasicek (1977) and CIR (1985) have gained prominence in the literature of derivative contract pricing; for example, CIR (1985), Jamshidian (1989), Rabinovitch (1989), Hull and White (1990) and Duan, Moreau and Sealey (1995).

The empirical research focussing on some particular models of the exponential–affine family is extensive. The existent literature can be loosely classified into four categories. The first approach uses proxies for the unobserved factors; for example, Marsh and Rosenfeld (1983), Chan, *et al.* (1992) and Daves and Ehrhardt (1993). The second approach uses a cross-sectional restriction implied by a term structure model to yield a reduced-form model.

Examples of this approach are Brown and Dybvig (1986), Titman and Torous (1989), and De Munnik and Schotman (1994). The third category, such as Gibbons and Ramaswamy (1993), involves the derivation of conditional moment restrictions and the application of the generalized method of moments for estimation. The fourth category is the application of the transformed data maximum likelihood method proposed in Duan (1994). Particular cases of this approach are found in Pearson and Sun (1994) and Chen and Scott (1993a). The method proposed in this paper can be regarded as a generalization of the transformed data method. The transformed data method uses a term structure model to define a one-to-one data transformation from the unobserved state variables to the observed bond yields. This method breaks down when the number of yield series is greater than the number of state variables, unless one is willing to impose *ad hoc* restrictions on the structure of the measurement errors.

The purpose of this study is to develop a unified framework in which all exponential–affine term structure models can be estimated with the simultaneous use of many bond yield series. We utilize a result of Duffie and Kan (1993) which establishes the sufficient and necessary conditions for the obtention of an exponential–affine term structure model. These conditions simply require the drift and variance functions of the underlying diffusion process to be affine in state variables. In this paper, we establish that the conditional mean and variance function of the state variables, over any discrete–time interval, must also be affine in state variables. This result for the discrete–time interval makes it possible to use the Kalman filter and the prediction–error decomposition to obtain an approximate quasi–maximum likelihood solution to the estimation problem for the entire class of exponential affine term structure models.

Special cases of our approach exist in the literature. To our knowledge, Pennacchi (1991) is the first study that uses a state-space formulation for estimating a term structure model of Gaussian nature. Chen and Scott (1993b) and Lund (1994) also propose the use of a Kalman filter-based method to estimate term structure model. When our method is applied to a Gaussian model, if measurement errors are also normally distributed, our method becomes a maximum likelihood estimation.

The proposed method is implemented for the yields on the U.S. Treasury debt securities with maturities: 1, 3, 6, and 9 months. The empirical study is carried out using monthly data for the period from July, 1964 to February, 1992. Three term structure models of the exponential affine family – Vasicek (1977), CIR (1985), and Chen and Scott (1992) – are analyzed. For the Vasicek (1977) and CIR (1985) models, the parameter estimates in most cases are statistically significant. Using an approximate robust Lagrange multiplier test, we find that all three models are strongly rejected.

To examine the statistical properties of the proposed method, a Monte Carlo study is carried out. The simulation study suggests that the method is reasonably reliable even for a sample size as small as 150.

2. A State-Space formulation for the exponential–affine Term Structure Models

Vasicek (1977) and CIR (1985) bond pricing models have been widely used in the term structure literature. Both models involve specifying a stochastic process for the unobserved instantaneous spot interest rate. While Vasicek specifies a continuous time Ornstein-Uhlenbeck process, CIR use a mean-reverting square root process. Multi–factor generalizations of the CIR model have been proposed by Chen and Scott (1992) and Longstaff and Schwartz (1992). Recently Duffie and Kan (1993) have provided a characterization for the class of exponential–affine term structure model which contains, amongst others, the aforementioned models as special cases.

The exponential–affine term structure model is a class of models in which the yields to maturity are affine functions in some abstract state variable vector X_t, which is assumed to obey the following dynamic:

$$dX_t = U(X_t; \Psi)dt + \Sigma(X_t; \Psi)dW_t, \tag{2.1}$$

where W_t is a $n \times 1$ vector of independent Wiener processes; Ψ denotes a $p \times 1$ vector containing the model parameters; $U(.)$ and $\Sigma(.)$ are $n \times 1$ and $n \times n$ functions regular enough so that equation (1) has a unique solution. The bond pricing formula for this class of model

can be generically expressed as:

$$D_t(X_t; \Psi, \tau) = A(\Psi, \tau) e^{-B(\Psi,\tau) X_t}, \tag{2.2}$$

where $D_t(X_t; \Psi, \tau)$ is the price at time t of a default–free discount bond with time to maturity τ; $A(\Psi, \tau)$ is a scalar function and $B(\Psi, \tau)$ is a $1 \times n$ vector function. The instantaneous interest rate is, as usual, defined as

$$r_t(X_t; \Psi) = -\lim_{\tau \to 0} \frac{\ln D_t(X_t; \Psi, \tau)}{\tau}. \tag{2.3}$$

Duffie and Kan (1993) have shown that $D(.)$ is generically exponential–affine, i.e. in the form of equation (2), if and only if $U(.)$, $\Sigma(.)\Sigma(.)'$ and $r_t(.)$ are affine in X_t. Moreover $A(.)$ and $B(.)$ in equation (2) can be obtained as the solutions to some ordinary differential equations (see Appendix A).

Let $R_t(X_t; \Psi, \tau)$ denote the time-t continuously compounded yield on a zero-coupon bond of maturity τ. The yield-to-maturity of this bond is given by:

$$R_t(X_t; \Psi, \tau) = -\frac{1}{\tau} \ln(D_t(X_t; \Psi, \tau)). \tag{2.4}$$

To deal with the estimation problem, it is reasonable to assume that the yields for different maturities are observed with errors of unknown magnitudes. Using the bond pricing formula in (2), the yield to maturity can be written, after the addition of a measurement error, as:

$$R_t(X_t; \Psi, \tau) = -\frac{1}{\tau} \ln(A(\Psi, \tau)) + \frac{1}{\tau} B(\Psi, \tau) X_t + \epsilon_t, \tag{2.5}$$

where ϵ_t is an error term with zero mean and standard deviation σ_ϵ. Note that ϵ_t need not be normally distributed.

Given that N bonds with different maturities are observed, the N corresponding yields can be stacked to obtain the following representation:

$$\begin{bmatrix} R_t(X_t; \Psi, \tau_1) \\ R_t(X_t; \Psi, \tau_2) \\ \vdots \\ R_t(X_t; \Psi, \tau_N) \end{bmatrix} = \begin{bmatrix} -\ln(A(\Psi, \tau_1))/\tau_1 \\ -\ln(A(\Psi, \tau_2))/\tau_2 \\ \vdots \\ -\ln(A(\Psi, \tau_N))/\tau_N \end{bmatrix} + \begin{bmatrix} (1/\tau_1) B(\Psi, \tau_1) \\ (1/\tau_2) B(\Psi, \tau_2) \\ \vdots \\ (1/\tau_N) B(\Psi, \tau_N) \end{bmatrix} X_t + \begin{bmatrix} \epsilon_{t,1} \\ \epsilon_{t,2} \\ \vdots \\ \epsilon_{t,N} \end{bmatrix}. \tag{2.6}$$

In terms of the state-space model, this equation is referred to as the measurement equation.

To obtain the transition equation for the state-space model, we need to derive expressions for the conditional mean and variance of the unobserved state variables process over a discrete time interval of length h. Define $m(X_t;\Psi,h) \equiv \mathrm{E}(\mathrm{X_{t+h}}|\mathrm{X_t})$ and $\Phi(X_t;\Psi,h) \equiv \mathrm{Var}(\mathrm{X_{t+h}}|\mathrm{X_t})$. The transition equation over a discrete time interval, with length equal to h units of time, can be written as:

$$X_{t+1} = m(X_t;\Psi,h) + \Phi(X_t;\Psi,h)^{1/2}\eta_{t+1} \tag{2.7}$$

where η_{t+1} is a $n \times 1$ vector of zero mean and unit variance error terms with $\Phi(X_t;\Psi,h)^{1/2}$ denoting the Cholesky decomposition of $\Phi(X_t;\Psi,h)$.

For a Gaussian state-space model, the Kalman filter provides an optimal solution to prediction, updating and evaluating the likelihood function. The Kalman filter recursion is a set of equations which allows an estimator to be updated once a new observation becomes available. The Kalman filter first forms an optimal predictor of the unobserved state variable vector given its previously estimated value. This prediction is obtained using the distribution of the unobserved state variables, conditional on the previous estimated values. These estimates for the unobserved state variables are then updated using the information provided by the observed variables. Prediction errors, obtained as a by-product of the Kalman filter, can then be used to evaluate the likelihood function.

When the state-space model is non-Gaussian, the Kalman filter can still be applied to obtain approximate first and second moments of the model and the resulting filter is quasi-optimal. The use of this quasi–optimal filter yields an approximate quasi–likelihood function with which parameter estimation can be carried out.

Given that $U(.)$ and $\Sigma(.)\Sigma(.)'$ are affine functions of X_t, $m(X_t;\Psi,h)$ and $\Phi(X_t;\Psi,h)$ must be affine in X_t (see Appendix B). Let $\mathcal{F}_t$ be the information set generated by the observations up to and including time t. That is, $\mathcal{F}_t$ is a filtration generated by $\{R_s(X_s;\Psi,\tau_1),\cdots,$ $R_s(X_s;\Psi,\tau_N)$; fors $\leq$ t$\}$. The conditional mean and variance are functions of the lagged unobserved state variables. This implies that $m(X_t;\Psi,h)$ and $\Phi(X_t;\Psi,h)$ are not measurable with respect to $\mathcal{F}_t$. However, their expected values conditional on $\mathcal{F}_t$ can easily be expressed.

Define $P_{t+1|t} \equiv \mathrm{Var}(\mathrm{X_{t+1}}|\mathcal{F}_\mathrm{t})$ and $P_t \equiv \mathrm{Var}(\mathrm{X_t}|\mathcal{F}_\mathrm{t})$. Because $m(.)$ is affine in X_t we have:

$$m(\hat{X}_t; \Psi, h) = a(\Psi, h) + b(\Psi, h)\hat{X}_t, \tag{2.8}$$

where $a(.)$ and $b(.)$ are $n \times 1$ and $n \times n$ matrices and $\hat{X}_t \equiv \mathrm{E}(\mathrm{X_t}|\mathcal{F}_\mathrm{t})$. To implement the quasi–optimal Kalman filter, one must derive a relationship between $P_{t+1|t}$ and P_t. By the law of iterated expectations and recognizing that $\Phi(X_t; \Psi, h)$ is affine in X_t and $\mathrm{Cov}(\mathrm{X_t}, \Phi(\mathrm{X_t}; \Psi, \mathrm{h})^{1/2}\eta_{\mathrm{t+1}}|\mathcal{F}_\mathrm{t}) = 0$ one can derive:

$$P_{t+1|t} = b(\Psi, h)P_t b(\Psi, h)' + \Phi(\hat{X}_t; \Psi, h). \tag{2.9}$$

If it were possible to compute $\hat{X}_t$, the conditional mean and variance of the system could be correctly specified. Using the measurement and transition equations described in (6) and (7), the standard Kalman filter recursion [1] could then be used to obtain a prediction-error decomposition to evaluate the quasi-likelihood function. The estimates obtained with this quasi–likelihood function would, according to Bollerslev and Wooldridge (1992), are consistent and asymptotically normal. Unfortunately, the linear Kalman filter cannot produce $\hat{X}_t$; rather it yields $\bar{X}_t$, that is the linear projection of X_t on the linear sub–space generated by the observed yields. The conditional mean and variance computed with the estimate $\bar{X}_t$ should thus be different from the true conditional mean and variance of the system. Nevertheless, this approximation can be expected to work well because it is linearly optimal. This approximation is needed because of the non–Gaussian nature of the problem, which can be likened to linearizing a non–linear function in the typical Kalman filtering applications.

Since $\bar{X}_t$ is computable, we use it to approximate $\hat{X}_t$ which yields a quasi-likelihood function in an approximate sense. This approximate quasi–likelihood function is used as if it were the correct quasi-likelihood function. Since the statistical properties of this procedure are theoretically unknown, Monte Carlo experiments are conducted in Section 4 to assess the quality of this procedure.

Let

$$L_T(\Psi; \mathbf{R}) = \sum_{t=1}^{T} l_t(\Psi; \mathbf{R_t}), \tag{2.10}$$

[1] A complete description of the Kalman filter recursion can be found in Harvey (1990).

where $l_t(\Psi; \mathbf{R_t})$ denotes the quasi-likelihood function at time t obtained from the approximate prediction-error decomposition with $\mathbf{R_t} = \{R_t(X_t; \Psi, \tau_1), \ldots, R_t(X_t; \Psi, \tau_N)\}'$ and $\mathbf{R} = (\mathbf{R_1}, \ldots, \mathbf{R_T})$. Following Bollerslev and Wooldridge (1992), it is reasonable to expect the parameter vector $\hat{\Psi}_T$ maximizing $L_T(\Psi; \mathbf{R})$ to be approximately consistent and asymptotically normal. This approximately consistent parameter should have the following approximate asymptotic distribution:

$$\sqrt{T}(\hat{\Psi}_T - \Psi_0) \sim N(0, \hat{F}_T^{-1} \hat{G}_T \hat{F}_T^{-1}), \tag{2.11}$$

where

$$\hat{F}_T = \frac{1}{T} \sum_{t=1}^{T} f_t(\hat{\Psi}_T; \mathbf{R_t}), \tag{2.12}$$

$$\hat{G}_T = \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \ln l_t(\hat{\Psi}_T; \mathbf{R_t})}{\partial \Psi}' \frac{\partial \ln l_t(\hat{\Psi}_T; \mathbf{R_t})}{\partial \Psi}, \tag{2.13}$$

with

$$f_t(\hat{\Psi}; \mathbf{R_t}) = \frac{\partial \mu_t'}{\partial \Psi} \Omega_t^{-1} \frac{\partial \mu_t}{\partial \Psi} + \frac{1}{2} \frac{\partial \Omega_t'}{\partial \Psi} [\Omega_t^{-1} \otimes \Omega_t^{-1}] \frac{\partial \Omega_t}{\partial \Psi},$$

where μ_t and Ω_t are the conditional mean and variance functions obtained from the linear Kalman filter and $\frac{\partial \ln l_t(\hat{\Psi}_T; \mathbf{R_t})}{\partial \Psi}$, $\frac{\partial \mu_t}{\partial \Psi}$ and $\frac{\partial \Omega_t}{\partial \Psi}$ are of dimension $1 \times p$, $N \times p$ and $N^2 \times p$ respectively

The asymptotic covariance matrix of the quasi-maximum likelihood estimator, commonly referred to as the robust covariance matrix, is in general not equal to the inverse of Fisher's information matrix, $-\lim_{T \to \infty} \hat{F}_T$. For Gaussian exponential–affine term structure models, the approximate quasi-likelihood function becomes the exact likelihood function if the measurement errors are normally distributed. This then implies that the asymptotic variance is $(-\lim_{T \to \infty} \hat{F}_T)^{-1}$ or $(\lim_{T \to \infty} \hat{G}_T)^{-1}$, a standard result.

To test the over–identification restrictions imposed by the theoretical model, an approximate robust Lagrange multiplier test can be constructed. Denote the unconstrained

measurement equation to be:

$$\begin{bmatrix} R_t(X_t;\Psi,\tau_1) \\ R_t(X_t;\Psi,\tau_2) \\ \vdots \\ R_t(X_t;\Psi,\tau_N) \end{bmatrix} = \begin{bmatrix} -\ln(A(\Psi,\tau_1))/\tau_1 + \alpha_1 \\ -\ln(A(\Psi,\tau_2))/\tau_2 + \alpha_2 \\ \vdots \\ -\ln(A(\Psi,\tau_N))/\tau_N + \alpha_N \end{bmatrix} + \begin{bmatrix} (1/\tau_1)B(\Psi,\tau_1) + \beta_1 \\ (1/\tau_2)B(\Psi,\tau_2) + \beta_2 \\ \vdots \\ (1/\tau_N)B(\Psi,\tau_N) + \beta_N \end{bmatrix} X_t + \begin{bmatrix} \epsilon_{t,1} \\ \epsilon_{t,2} \\ \vdots \\ \epsilon_{t,N} \end{bmatrix}. \tag{2.14}$$

where α_1 to α_N are scalar parameters; β_1 to β_N are parameter vectors of dimension $1 \times n$.

This specification for the unconstrained measurement equation is over–parameterized. First, the exponential–affine term structure model contains risk premium parameters that do not appear in the transition equation. Specifically, risk premium parameters appear in functions $A(.)$ and/or $B(.)$ of the measurement equation. These risk premium parameters complicate the matter because the unconstrained model cannot be expressed solely in terms of the parameters α_i, β_i and those in the transition equation. Second, since X_t is an unobserved process, its location and scale parameters are indeterminate. This specification under the alternative hypothesis in equation (14) is thus under–identified.

To perform a Lagrange multiplier test of the null hypothesis, H_0: $\alpha_i = 0, \beta_i = 0$ for all i, one must deal with the over–parameterization problem by appropriately removing some parameters from the parameter space. More specifically, n parameters must be removed to account for the risk premia, and $\frac{1}{2}n(n+1)+n$ parameters removed for the alternative model identification (see Harvey (1990), section 8.5.1). The decision about which parameters can be conveniently removed turns out to be model–specific. We thus address this issue when the specific bond pricing models are discussed in Section 3.

The approximate robust Lagrange multiplier test proceeds as follows. Denote by θ the parameter vector under the alternative, that is let $\theta' = \{\Psi', \phi'\}$ where ϕ is a parameter vector of dimension $N(n+1) - \frac{1}{2}n(n+1) - 2n$. The parameter vector Ψ is a subset of $\{\alpha_i, \beta_i\}$ after removing an appropriate number of parameters. The null hypothesis can be stated as:

$$H_0: \quad \phi = 0.$$

A result from White (1982) and Domowitz and White (1982) can be used to construct the

approximate robust LM test statistic as follows:

$$\varepsilon_{LM} = \hat{S}_{1T}\hat{F}_T^{11}\hat{C}_{11}^{-1}\hat{F}_T^{11}\hat{S}_{1T}' \rightarrow \chi^2_{\{N(n+1)-\frac{1}{2}n(n+1)-2n\}}, \tag{2.15}$$

where $\hat{F}_T^{11}$ is the partitioned inverse corresponding to ϕ; $\hat{C} = \hat{F}_T^{-1}\hat{G}_T\hat{F}_T^{-1}$; and $\hat{C}_{11}$ is the submatrix of $\hat{C}$ corresponding to ϕ; $\hat{S}_T = \frac{\partial L_T(\theta;\mathbf{R})}{\partial\theta}$ and $\hat{S}_{1T}$ is the subvector of $\hat{S}_T$ corresponding to ϕ.

3. Empirical results

The data set used in this study consists of four monthly yield series for the U.S. Treasury debt securities with maturities: 1, 3, 6 and 9 months taken from the Fama–Bliss data file. All interest rates are expressed on an annualized continuously compounded basis. These data series cover the period from July, 1964 to February, 1992, totalling 332 time series observations each. Table 1 reports summary statistics for these data series. Many empirical studies, for example, Hamilton (1988) and Spindt and Tarhan (1987), have found that the shift in the Federal Reserve monetary policy from October, 1979 to October, 1982, caused a structural break in the interest rate process. We thus report the results for the entire sample period as well as for two subperiods: July, 1964 to October, 1979 and October, 1982 to February, 1992.

The basic unit of time in this study is set equal to one year so that the maturities for all yields are stated in terms of the number of years and the parameter estimates can be interpreted as annualized values. Since the data frequency is monthly, the length of the discrete sampling interval, h, equals 1/12. The numerical optimization routine used in this study is the quadratic hill-climbing algorithm of Goldfeld, Quandt and Trotter (1966). The convergence criterion, based on the maximum absolute difference in both parameter and functional values between two successive iterations, is set to 0.0001. For each model examined below, the approximate Kalman filter recursion is initialized with the stationary mean and variance of the unobserved state–variable(s).

3.1. The Gaussian case: Vasicek model

For the Vasicek (1977) model, the unobserved state variable is the instantaneous interest rate which has the following specification:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dz_t, \tag{3.1}$$

where z_t is a Wiener process; θ is the long–run average of the instantaneous spot interest rate; $\kappa \geq 0$ is the mean–reverting intensity at which the process returns to its long–run mean; and $\sigma > 0$ is the volatility parameter of the process.

In terms of this model, the functional forms for $A(.)$, $B(.), a(.), b(.)$ and $\Phi(.)$ are given by:

$$\begin{aligned}
A(\Psi,\tau) &= e^{[\gamma(B(\Psi,\tau)-\tau)-\frac{\sigma^2 B^2(\Psi,\tau)}{4\kappa}]}, && (3.2)\\
B(\Psi,\tau) &= \frac{1}{\kappa}\left[1-e^{-\kappa\tau}\right], && (3.3)\\
\gamma &= \theta + \frac{\sigma\lambda}{\kappa} - \frac{\sigma^2}{2\kappa^2}, && (3.4)\\
a(\Psi,h) &= \theta(1-e^{-\kappa h}), && (3.5)\\
b(\Psi,h) &= e^{-\kappa h}, && (3.6)\\
\Phi(X_t;\Psi,h) &= \frac{\sigma^2}{2\kappa}(1-e^{-2\kappa h}), && (3.7)
\end{aligned}$$

where λ is the risk premium parameter. In this model, $\lambda > 0$ implies a positive premium for bond prices.

We assume a diagonal covariance structure for the measurement errors. Its elements are denoted by $\sigma^2_{\epsilon_1}$, $\sigma^2_{\epsilon_3}$, $\sigma^2_{\epsilon_6}$, $\sigma^2_{\epsilon_9}$. The second column of Tables 2A, B and C, present the empirical results for the Vasicek model. Parameter estimates are reported along with their robust standard errors in parentheses. In Table 2A, the results for the entire sample period are reported. The long–run average interest rate, θ, the mean-reverting parameter κ and σ_{ϵ_6} are not significantly different from zero at the usual significance level. All other parameters are significantly different from zero. For the two sub–periods, reported in Tables 2B and 2C, all parameters are statistically significant except for κ and σ_{ϵ_6} in the first and second sub–period.

When compared to the full sample estimates, the estimates for the pre–October, 1979 period as well as those for the post–October, 1982 period indicate a smaller instantaneous volatility, σ. This result supports the finding in the literature that the shift in the Federal Reserve monetary policy from October, 1979 to October, 1982 caused a structural break in the interest rate process.

The Lagrange multiplier test for the restrictions imposed by the Vasicek model on the coefficients of the measurement equation is performed using the specification described at the end of Section 2. In order to make the factor model under the alternative hypothesis identified, α_1 and β_1 are conveniently removed from the unconstrained parameter space. For the Vasicek model, only $A(.)$ are functions of the risk premium parameter λ. The parameter α_2 can be conveniently regarded as an extra parameter. It is thus also discarded from the unconstrained model parameter space. As a result the degrees of freedom for this test on four yield series equals five. For the whole as well as sub-samples, the null hypothesis is strongly rejected as indicated at the bottom portion of Tables 2A, B and C.

3.2. The non-Gaussian case: Chen and Scott and CIR models

The Chen and Scott (1992) model allows for several independent state variables with the following dynamic:

$$dx_{i,t} = \kappa_i(\theta_i - x_{i,t})dt + \sigma_i\sqrt{x_{i,t}}dz_{i,t}, \tag{3.8}$$

for $i = 1, ..., k$ and $z_{i,t}$ are independent Wiener processes.

In this section, the one and two–factor version of their model are analyzed. The one–factor version of their model correspond to CIR (1985) model, where the unobserved factor is interpreted as the instantaneous interest rate.

For the two–factor model, the functional forms for $A(.)$ and $B(.)$ are given by:

$$A(\Psi, \tau) = A_1(\Psi, \tau)A_2(\Psi, \tau), \tag{3.9}$$

$$B(\Psi, \tau) = [\, B_1(\Psi, \tau) \;, \quad B_2(\Psi, \tau)\,], \tag{3.10}$$

where $A_i(\Psi,\tau)$ and $B_i(\Psi,\tau)$ for $i=1$ and 2 have the following specification:

$$A_i(\Psi,\tau) = \left[\frac{2\gamma_i e^{[(\kappa_i+\lambda_i+\gamma_i)\tau]/2}}{(\kappa_i+\lambda_i+\gamma_i)(e^{\gamma_i\tau}-1)+2\gamma_i}\right]^{2\kappa_i\theta_i/\sigma_i^2}, \tag{3.11}$$

$$B_i(\Psi,\tau) = \frac{2(e^{\gamma_i\tau}-1)}{(\kappa_i+\lambda_i+\gamma_i)(e^{\gamma_i\tau}-1)+2\gamma_i}, \tag{3.12}$$

with

$$\gamma_i = \sqrt{(\kappa_i+\lambda_i)^2+2\sigma_i^2}. \tag{3.13}$$

The functional forms for $a(.), b(.)$ and $\Phi(.)$ are given by:

$$a(\Psi,h) = \begin{bmatrix}\theta_1(1-e^{-\kappa_1 h})\\ \theta_2(1-e^{-\kappa_2 h})\end{bmatrix}, \tag{3.14}$$

$$b(\Psi,h) = \begin{bmatrix}e^{-\kappa_1 h} & 0\\ 0 & e^{-\kappa_2 h},\end{bmatrix}, \tag{3.15}$$

$$\Phi(X_t;\Psi,h) = \begin{bmatrix}\Phi_1(x_{1,t};\Psi,h) & 0\\ 0 & \Phi_2(x_{2,t};\Psi,h)\end{bmatrix}, \tag{3.16}$$

with

$$\Phi_i(x_t;\Psi,h) = x_t\frac{\sigma_i^2}{\kappa_i}(e^{-\kappa_i h}-e^{-2\kappa_i h}) + \theta_i\frac{\sigma_i^2}{2\kappa_i}(1-e^{-\kappa_i h})^2 \tag{3.17}$$

where λ_1 and λ_2 are the risk premia parameters. In this model, $\lambda_i < 0$ implies a positive premium in bond prices for factor i. The CIR (1985) model is a special case of this model, and is obtained by using the top left elements of the matrices defined above.

The estimation results for the CIR (1985) model are reported in the third column of Tables 2A, B and C. These tables contain parameter estimates along with their corresponding robust standard errors in parentheses. Except for κ in the second sub–sample and some measurement errors standard deviation, all estimates are significantly different from zero at the 5% level in all samples. In all cases, the unobserved instantaneous interest rate is found to exhibit mean-reverting behavior, i.e., $\kappa > 0$. The magnitude of the measurement errors are generally small but significantly different from zero in many cases.

Comparing the estimates for the entire sample with the ones for the pre–October, 1979 and the post–October, 1982 periods reveals that a smaller volatility scale parameter, σ, prevails for these two subperiods. These results are consistent with the previous findings

that the shift in the Federal Reserve monetary policy from October, 1979 to October, 1982 caused a structural break in the interest rate process. The mean-reverting property of the interest rate process again prevails for these two subperiods.

The Lagrange multiplier test of the restrictions imposed by the CIR model on the coefficients of the measurement equation is performed using the specification described at the end of Section 2. For the CIR model, both $A(.)$ and $B(.)$ are functions of the risk premium parameter λ. As with the Vasicek model, we remove α_1, α_2 and β_1 from the unconstrained parameter space to make the model identified under the alternative specification. As a result, the degrees of freedom for this test equals five. The null hypothesis is strongly rejected in the three samples as shown by the P-values of the Chi-square tests.

The estimation results for the Chen and Scott (1992) model are reported in the fourth column of Tables 2A, B and C. These tables contain parameter estimates along with their corresponding robust standard errors in parentheses. For the entire sample period and the first sub–sample, all parameters are significantly different from zero except for θ_2, κ_2 and λ_2. For the second sub-samples only θ_2 and κ_2 are not significantly different from zero.

The most striking feature of these estimates are the small values for θ_2 and κ_2. With these values, $\sigma_2^2 > 2\kappa_2\theta_2$ and the second factor can reach zero with a positive probability (see CIR (1985)). A small value for κ_2 also suggest that the second factor is close to being non stationary. The resulting test statistics might therefore differ from the approximate asymptotic normal distribution and should be interpreted with care.

Comparing the estimates for the entire sample with the ones for the pre–October, 1979 and post–October, 1982 periods still points to a structural instability even when the two–factor model is used. This result is once again consistent with the previous findings that the shift in the Federal Reserve monetary policy from October, 1979 to October, 1982 caused a structural break in the interest rate process.

Chen and Scott (1993b) report the estimation results for the Chen and Scott (1992) two-factor model. Their estimates are consistent with our findings. More specifically, the estimates for the long–run mean and mean reversion parameters of the second factor are small, suggesting a nearly integrated second factor.

In the Chen and Scott (1992) model, both $A(.)$ and $B(.)$ are functions of the risk premium parameters. To perform the robust Lagrange multiplier test described at the end of Section 2, β_N, which is a $1 \times n$ vector, can be conveniently removed from the parameter space to account for the risk premium parameters. To ensure that the factor model under the alternative specification is identified, α_1, α_2, β_1 and the second entry of β_2 are removed from the unconstraint parameter space. The resulting degrees of freedom for this Chi–square test also equals five. In all samples the null hypothesis is strongly rejected as indicated by the P-values of the Chi-square tests.

4. A Monte Carlo analysis

To assess the statistical properties of the proposed method, we conduct Monte Carlo experiments for the sample size of 150, 350 time-series observations with the simultaneous use of 1-, 3-, 6- and 9-month maturities. These sample sizes are chosen because they roughly correspond to the sub and whole sample size of the data sets analyzed in the preceding section.

To perform Monte Carlo experiments, the unobserved state variables of a given model must first be simulated. For the Vasicek (1977) model, the state variable follows an Ornstein–Uhlenbeck process which can be obtained using the exact conditional distribution. More formally, denote r_t^* to be the simulated value of the process at time t. Using the transition density of this process, r_t^* is obtained by:

$$r_t^* - r_{t-h}^* = \theta + (r_{t-h}^* - \theta)\ e^{-\kappa h} + \sqrt{\phi}\epsilon_t,$$

where ϵ_t is $N(0,1)$ with $r_0^* = \theta$ and $\phi = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa h})$ and h set to $1/12$.

For the square–root process in CIR (1985), the exact conditional distribution is also used to obtain the simulated values. Let x_t denote the value of an unobserved state variable following a square–root process. The conditional distribution function for x_t is the non–central chi–square, $\chi^2[\ 2cx_t;\ 2d+2, 2w]$, with $2d+2$ degrees of freedom and parameter of non–centrality $2w$ where

$$c = \frac{2\kappa}{\sigma^2(1 - e^{-\kappa h})},$$

$$d = \frac{2\kappa\theta}{\sigma^2} - 1,$$
$$w = cx_{t-h}e^{-\kappa h}.$$

As discussed in Johnson and Kotz (1970), the non–central Chi-square distribution can be expressed as a mixture of central Chi-squares with degrees of freedom proportional to random variates from a Poisson distribution. This property is used to simulate the process. More precisely, x_t^*, the simulated value of the process at time t, is obtained by the following steps:

1. Simulate the degrees of freedom of the central chi–square using

$$df = 2d + 2 + 2j,$$

where j is a Poisson random variate [2] with mean $w = cx_{t-h}^* e^{-\kappa h}$. The time interval h is set to $1/12$ and $x_0^* = \theta$.

2. Let g denote the random variate drawn from the central chi–square with df degrees of freedom. This random variate is obtained from a gamma $(\frac{df}{2}, 2)$ random variate [3] where $\frac{df}{2}$ is the shape parameter and 2 is the scale parameter.

3. Compute $x_t^* = \frac{g}{2c}$.

Using the simulated time series for either the Ornstein–Uhlenbeck or the square-root process, equation (5) is then used to compute the simulated yields with independent normally distributed measurement errors. The Monte Carlo simulation results for discretely sampled time-series are reported for the Gaussian case (Vasicek (1977)) and the non–Gaussian case (CIR (1985)). The results are obtained using 500 repetitions.

Rows one, two, three and four of Tables 3 and 4 report the true parameter values, the medians, the means and the standard deviations of the parameter estimates. The remaining four rows report the probability that the true parameter lies in the $\alpha\%$ confidence interval. This probability is referred to as the $\alpha\%$ coverage rate. For example, to obtain 95% coverage rate, $\text{Prob}\{|\Psi_\text{i} - \hat{\Psi}_\text{iT}| < 1.96 \text{ s.e.}(\hat{\Psi}_\text{iT})\}$ is to be computed, where Ψ_i denotes the ith

[2] The algorithm used to simulate the Poisson random variate is taken from Devroye (1981).
[3] The algorithm used to simulate the gamma random variate is taken from Devroye (1986).

parameter of the model and s.e.($\hat{\Psi}_{iT}$) represents the estimated standard error for the ith parameter estimator. The bottom row of each panel reports the 95% coverage rate for the over–identification test statistic. This value indicates whether the test is biased or not.

The Monte Carlo simulation results for the Gaussian case (Vasicek (1977) model) are presented in Table 3A. The true parameter values are taken from the estimates, after rounding, in the empirical section. The estimated values are exceedingly close to their corresponding true values for all parameters except θ and κ. The magnitude of the bias for these two parameters is small as a percentage of their corresponding true values. The coverage rates of the parameters indicate some departure from the asymptotic distribution, especially when a smaller sample size is used. A possible explanation for this departure is the small value given to κ. As κ approaches zero, the unobserved factor and the yield series become non–stationary.

To investigate further, additional Monte Carlo experiments with 350 observations and different parameter values are are reported in the third panel of Table 3B. The top panel of this table reports the results for the case of a larger value for κ, the mean–reverting parameter. When $\kappa = .5$, the biases nearly vanish, and the asymptotic distribution becomes a better approximation to the small sample distribution. The bottom panel of this table reports the results for the case of a smaller mean–reverting parameter value. When its value is set to 0.01, the coverage rates become similar to those reported in the second panel of Table 3A. This result suggests that the magnitude of the mean reversion parameter plays an important role in determining the quality of the estimator when using a finite sample size. This result is not surprising. The mean–reverting parameter κ is a transformation of the first order autoregressive parameter. It is well known in the time–series literature that the standard asymptotic properties of the estimators do not hold in the presence of a unit root.

The 95% coverage rate of the Lagrange multiplier test indicates a slight departure from the asymptotic distribution for all Monte Carlo experiments in Table 3. The test appears more conservative, meaning that the null hypothesis is rejected slightly more than 5% of the time.

The Monte Carlo simulation results for the non–Gaussian CIR (1985) model are presented

in Table 4A. The true parameter values are taken from the estimates, after rounding, in the preceding empirical section. The estimates are close to their corresponding true values for all parameters. As with the Vasicek (1977) model, θ and κ are slightly biased. However, the magnitude of the bias is small as a percentage of the true parameter value. Overall the coverage rates of the parameters indicate that the asymptotic distribution to the quasi–maximum likelihood estimator is a good approximation of the small sample distribution. In order to assess the impact of a nearly integrated factor, additional Monte–Carlo experiments with 350 observations and different parameter values are reported in Table 4B. The top portion of this table reports the results for a larger value of κ. As in the Gaussian case with $\kappa = .5$, the biases nearly vanish, and the approximate asymptotic distribution becomes a better approximation to the small sample distribution. The bottom panel of this table reports the results for the same parameter values with the exception of κ, which is now set to 0.01, a value closer to zero. With these parameter values, the estimates for θ and κ are biased and their coverage rates are poorly approximated by the asymptotic distribution.

Again, as with the Gaussian case, the 95% coverage rate of the Lagrange multiplier test also indicates some departure from the asymptotic distribution in all Monte Carlo experiments in Table 4A. The test is typically more conservative, meaning that the null hypothesis is rejected more than 5% of the time. Interestingly, the case with a smaller value for κ in the bottom panel of Table 4B actually yields a better result for the Lagrange multiplier test.

Overall, these results suggest that the approximate quasi–maximum likelihood estimator, the CIR case, and the maximum likelihood estimator, the Vasicek case, tend to behave in a similar fashion. Both estimators are sensitive to a nearly integrated factor, and both estimators tend to slightly over–reject the null hypothesis in the Lagrange multiplier test. By and large, the finite sample properties of the approximate quasi–maximum likelihood estimator are reasonably described by their asymptotic equivalents.

5. Conclusion

In this article, a unified state-space formulation is developed for estimating term structure models of the exponential–affine family. The method allows for measurement errors in the observed yields to maturity, and is therefore useful for simultaneous estimation using yields on many bonds with different maturities. The quasi-optimal Kalman filtering approach can be useful for implementing derivative asset pricing models that are based on these exponential–affine term structure models. This estimation method is able to produce parameter estimates as well as a linearly filtered estimate of the unobserved state variables. Since the parameter estimates and the filtered state variables can be obtained by using a set of yields that cover a desirable maturity spectrum, its application to derivative asset pricing is likely to be less influenced by the measurement error in any given yield series.

A Monte Carlo study indicates that the proposed method is an adequate procedure. The finite sample properties of the approximate quasi–maximum likelihood estimator are reasonably approximated by the asymptotic distribution presented in this paper.

Three special cases of the exponential–affine family are used to examine the estimation method. The empirical results are, in some instances, supportive of the properties typical of the exponential–affine term structure models; for example, the mean-reverting property of the interest rate process. This, however, does not suggest that the three exponential–affine term structure models analyzed in this paper are good descriptions of the bond yield behaviour. In fact the results from using four different maturities cast doubts as to whether these models can be applied to yields covering a large maturity spectrum. Using a robust Lagrange multiplier test, the three exponential affine models are strongly rejected. These results suggest that future research is needed in order to find a better specification within or beyond the exponential–affine family.

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Appendix A

Differential equations for $A(.)$ and $B(.)$ functions.

As shown in Duffie and Kan (1993), the solutions for the $A(.)$ and $B(.)$ functions can be found by solving the following differential equations:

$$-\frac{\partial B(\Psi,\tau)}{\partial t} + \mathcal{B}(B(\Psi,\tau)) = 0, \quad B(\Psi,0) = 0,$$
$$-\frac{\partial A(\Psi,\tau)}{\partial t} + \mathcal{A}(B(\Psi,\tau)) = 0, \quad A(\Psi,0) = 0,$$

with $-\frac{\partial B_i(\Psi,\tau)}{\partial t} + \mathcal{B}_i(B(\Psi,\tau))$ denoting the coefficients of $X_{i,t}$ and $-\frac{\partial A(\Psi,\tau)}{\partial t} + \mathcal{B}(B(\Psi,\tau))$ denoting the term not involving X_t in the following equation:

$$r_t(X_t;\Psi) = -\frac{\partial A(\Psi,\tau)}{\partial t} - \frac{\partial B(\Psi,\tau)}{\partial t}X_t + B(\Psi,\tau)U(X_t;\Psi)$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} B_i(\Psi,\tau)B_j(\Psi,\tau)\Sigma_i(X_t;\Psi)\Sigma_j(X_t;\Psi)',$$

where $\Sigma_i(.)$ is the ith line of matrix $\Sigma(.)$.

Appendix B

Theorem: Assume that the $n \times 1$ vector X_t obeys the following dynamic:

$$dX_t = U(X_t;\Psi)dt + \Sigma(X_t;\Psi)dW_t.$$

Suppose that $U(X_t;\Psi)$ and $\Sigma(X_t;\Psi)\Sigma(X_t;\Psi)'$ are affine functions of X_t so that $U(X_t;\Psi)$ can be written as $G + KX_t$ with G and K being matrices of dimension $n \times 1$ and $n \times n$. The mean and variance of X_{t+h}, conditional on X_t, are affine functions of X_t if K is diagonable.

Remark: A matrix is diagonable if all of its eigenvalues are distinct. The assumption of diagonability does not involve an appreciable loss of generality. Since the eigenvalues of a

matrix are continuous functions of its elements, if the matrix K has multiple eigenvalues, a slight alteration to any element produces a neighbouring system with distinct roots that, for all practical purposes, is the same as the original system. Therefore the statement of the theorem is a generic result.

Proof:

For notational convenience, let $\Sigma(X_t; \Psi) = \Sigma_t$. The integral representation for X_{t+h} can be written as:

$$X_{t+h} = X_t + \int_t^{t+h} [G + KX_s]ds + \int_t^{t+h} \Sigma_s dW_s.$$

Denote the eigenvalue decomposition of K, usually a non–symmetric matrix, by QkQ^{-1} where $Q^{-1}Q = I$ and k a square diagonal matrix. Premultiplying the above equation by Q^{-1} yields:

$$Y_{t+h} = Y_t + \int_t^{t+h} [g + kY_s]ds + \int_t^{t+h} \sigma_s dW_s,$$

where $Y_t = Q^{-1}X_t$, $g = Q^{-1}G$, and $\sigma_t = Q^{-1}\Sigma_t$. Taking conditional expectation gives rise to:

$$\mathrm{E}(Y_{t+h}|Y_t) = Y_t + \int_t^{t+h} [g + k\mathrm{E}(Y_s|Y_t)]ds.$$

The solution to this integral equation is:

$$\mathrm{E}(Y_{t+h}|Y_t) = e^{kh}Y_t - (e^{kh} - I)k^{-1}g.$$

The solution for $\mathrm{E}(X_{t+h}|X_t)$ can be recovered from the above solution which is an affine function of X_t.

The conditional variance of Y_t can be computed using:

$$\mathrm{Var}(Y_{t+h}|Y_t) = \mathrm{E}(Y_{t+h}Y'_{t+h}|Y_t) - \mathrm{E}(Y_{t+h}|Y_t)\mathrm{E}(Y_{t+h}|Y_t)'.$$

The elements of $\mathrm{E}(Y_{t+h}Y'_{t+h}|Y_t)$ can be computed as follows. A $[\frac{(n+1)n}{2}] \times 1$ vector f_t is formed

by stacking the individual elements of the upper triangle of $Y_{t+h}Y'_{t+h}$. That is let:

$$f_t = \begin{bmatrix} Y_{1,t}Y_{1,t} \\ Y_{1,t}Y_{2,t} \\ \vdots \\ Y_{1,t}Y_{n,t} \\ Y_{2,t}Y_{2,t} \\ Y_{2,t}Y_{3,t} \\ \vdots \\ Y_{n,t}Y_{n,t} \end{bmatrix} = \begin{bmatrix} f_t^{1,1} \\ f_t^{1,2} \\ \vdots \\ f_t^{1,n} \\ f_t^{2,2} \\ f_t^{2,3} \\ \vdots \\ f_t^{n,n} \end{bmatrix}.$$

Using Ito's lemma on f_t yields:

$$f_{t+h} = f_t + \int_t^{t+h} \left[\frac{\partial f_s}{\partial Y_s} u_s + \mathcal{D}_s ds \right] + \int_t^{t+h} \sigma_s \frac{\partial f_s}{\partial Y_s} dW_s,$$

where $u_t = g + kY_t$ with k being the diagonal matrix defined earlier. Moreover,

$$\frac{\partial f_t}{\partial Y_t} = \begin{bmatrix} \frac{\partial f_t^{1,1}}{\partial Y_{1,t}} & \frac{\partial f_t^{1,1}}{\partial Y_{2,t}} & \cdots & \frac{\partial f_t^{1,1}}{\partial Y_{n,t}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_t^{n,n}}{\partial Y_{1,t}} & \frac{\partial f_t^{n,n}}{\partial Y_{2,t}} & \cdots & \frac{\partial f_t^{n,n}}{\partial Y_{n,t}} \end{bmatrix},$$

$$\mathcal{D}_t = \frac{1}{2} \begin{bmatrix} \operatorname{tr}(\sigma_t \sigma'_t \frac{\partial^2 f_t^{1,1}}{\partial Y_t \partial Y'_t}) \\ \vdots \\ \operatorname{tr}(\sigma_t \sigma'_t \frac{\partial^2 f_t^{n,n}}{\partial Y_t \partial Y'_t}) \end{bmatrix},$$

and

$$\frac{\partial^2 f_t^{i,j}}{\partial Y_t \partial Y'_t} = \begin{bmatrix} \frac{\partial^2 f_t^{i,j}}{\partial Y_{1,t} \partial Y_{1,t}} & \cdots & \frac{\partial^2 f_t^{i,j}}{\partial Y_{1,t} \partial Y_{n,t}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f_t^{i,j}}{\partial Y_{n,t} \partial Y_{1,t}} & \cdots & \frac{\partial^2 f_t^{i,j}}{\partial Y_{n,t} \partial Y_{n,t}} \end{bmatrix}.$$

Since u_t is affine in Y_t, we can write:

$$\frac{\partial f_t}{\partial Y_t} u_t = pY_t + qf_t,$$

where p is a matrix of dimension $[\frac{(n+1)n}{2}] \times n$ and q a diagonal matrix of dimension $[\frac{(n+1)n}{2}] \times$

$[\frac{(n+1)n}{2}]$ given by:

$$q = \begin{bmatrix} k_{1,1}+k_{1,1} & 0 & \dots & 0 \\ 0 & k_{1,1}+k_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & k_{n,n}+k_{n,n} \end{bmatrix},$$

where $k_{i,j}$ is element i,j of matrix k. Since $\sigma_t\sigma_t'$ is affine in Y_t, it follows that:

$$\mathcal{D}_t = w + vY_t,$$

where w and v are matrices of dimension $[\frac{(n+1)n}{2}] \times 1$ and $[\frac{(n+1)n}{2}] \times n$, respectively. The stochastic integral equation can thus be rewritten as:

$$f_{t+h} = f_t + \int_t^{t+h} [pY_s + qf_s + w + vY_s]ds + \int_t^{t+h} \sigma_s \frac{\partial f_s}{\partial Y_s} dW_s.$$

Taking conditional expectation yields:

$$\mathrm{E}(f_{t+h}|Y_t) = f_t + \int_t^{t+h} [z_s + q\mathrm{E}(f_s|Y_t)]ds,$$

where $z_s = w + (p+v)\mathrm{E}(Y_s|Y_t)$. Clearly, z_s is a function of s and is an affine function of Y_t. The solution to this integral equation is

$$\mathrm{E}(f_{t+h}|Y_t) = e^{qh}f_t + \int_t^{t+h} e^{q(t+h-s)} z_s ds.$$

As this solution shows, the only non–affine element, in terms of Y_t, for $\mathrm{E}(Y_{t+h}Y_{t+h}'|Y_t)$ is given by

$$e^{qh}f_t.$$

Straightforward computations for $\mathrm{E}(Y_{t+h}|Y_t)\mathrm{E}(Y_{t+h}|Y_t)'$, shows that the only non–affine element is given by

$$e^{kh}Y_tY_t'e^{kh}.$$

The upper triangular elements of this matrix, when properly staked up, are precisely the elements of $e^{qh}f_t$. Therefore $\mathrm{Var}(Y_{t+h}|Y_t)$ is affine in Y_t. Converting Y_t back to X_t gives rise to the result that $\mathrm{Var}(X_{t+h}|X_t)$ is affine in X_t.

Table 1
Summary statistics

July 1964 to February 1992

maturity	mean	std. dev.	auto.
1 month	.0666	.0260	.9483
3 months	.0702	.0266	.9637
6 months	.0725	.0265	.9656
9 months	.0740	.0262	.9652

July 1964 to October 1979

maturity	mean	std. dev.	auto.
1 month	.0547	.0167	.9276
3 months	.0574	.0165	.9329
6 months	.0600	.0164	.9380
9 months	.0617	.0164	.9391

October 1982 to February 1992

maturity	mean	std. dev.	auto.
1 month	.0695	.0155	.9033
3 months	.0734	.0152	.9468
6 months	.0754	.0156	.9443
9 months	.0768	.0158	.9449

Table 2A
Estimation results for the Vasicek (1977), CIR (1985) and Chen and Scott (1992) models with monthly observations on 1, 3, 6 and 9 months yield series
July 1964 to February 1992

Parameters	Model		
	Vasicek	CIR	Chen and Scott
θ_1	0.0486	0.0633	0.0285
	(0.0350)	(0.0111)	(0.0015)
κ_1	0.0561	0.3189	5.5699
	(0.0351)	(0.0665)	(0.6103)
σ_1	0.0225	0.0750	0.1481
	(0.0022)	(0.0052)	(0.0183)
λ_1	0.8039	-0.2715	-1.5747
	(0.0979)	(0.0572)	(0.2232)
θ_2	–	–	1.2e-7
	–	–	(2.4e-6)
κ_2	–	–	0.0006
	–	–	(0.1146)
σ_2	–	–	0.1052
	–	–	(0.0074)
λ_2	–	–	-0.0449
	–	–	(0.1307)
σ_{ϵ_1}	0.0058	0.0058	0.0041
	(0.0004)	(0.0004)	(0.0003)
σ_{ϵ_3}	0.0025	0.0025	0.0013
	(0.0002)	(0.0002)	(0.0001)
σ_{ϵ_6}	4.23e-8	3.8e-9	0.0004
	(1.0662)	(15.010)	(0.0002)
σ_{ϵ_9}	0.0018	0.0018	0.0013
	(0.0001)	(0.0001)	(0.0001)
Chi-square	115.21	146.98	48.756
P-value	0.0000	0.0000	0.0000
df	5	5	5

Table 2B
Estimation results for the Vasicek (1977), CIR (1985) and Chen and Scott (1992) models with monthly observations on 1, 3, 6 and 9 months yield series
July 1964 to October 1979

Parameters	Model		
	Vasicek	CIR	Chen and Scott
θ_1	0.0626	0.0602	0.0265
	(0.0316)	(0.0106)	(0.0024)
κ_1	0.0591	0.3628	5.3813
	(0.0428)	(0.0737)	(0.6536)
σ_1	0.0153	0.0637	0.1068
	(0.0013)	(0.0045)	(0.0160)
λ_1	1.2520	-0.3205	-1.4781
	(0.1449)	(0.0650)	(0.2339)
θ_2	–	–	1.3e-7
	–	–	(0.0002)
κ_2	–	–	0.0001
	–	–	(0.1408)
σ_2	–	–	0.0926
	–	–	(0.0074)
λ_2	–	–	-0.1567
	–	–	(0.1556)
σ_{ϵ_1}	0.0034	0.0034	0.0019
	(0.0002)	(0.0002)	(0.0002)
σ_{ϵ_3}	0.0020	0.0020	0.0012
	(0.0002)	(0.0002)	(0.0002)
σ_{ϵ_6}	0.0003	0.0002	0.0006
	(0.0004)	(0.0004)	(0.0001)
σ_{ϵ_9}	0.0018	0.0018	0.0014
	(0.0001)	(0.0001)	(0.0001)
Chi-square	65.123	91.616	25.433
P-value	0.0000	0.0000	0.0001
df	5	5	5

Table 2C
Estimation results for the Vasicek (1977), CIR (1985) and Chen and Scott (1992) models with monthly observations on 1, 3, 6 and 9 months yield series
October 1983 to February 1992

Parameters	Model		
	Vasicek	CIR	Chen and Scott
θ_1	0.0609	0.0606	0.0352
	(0.0302)	(0.0181)	(0.0015)
κ_1	0.0094	0.0791	5.0808
	(0.0096)	(0.0637)	(0.8336)
σ_1	0.0131	0.0467	0.1152
	(0.0010)	(0.0036)	(0.0196)
λ_1	1.0812	-0.1998	-0.8669
	(0.0955)	(0.0243)	(0.2226)
θ_2	–	–	1.6e-5
	–	–	(0.0015)
κ_2	–	–	2.2e-5
	–	–	(0.0020)
σ_2	–	–	0.0764
	–	–	(0.0081)
λ_2	–	–	-0.1338
	–	–	(0.0463)
σ_{ϵ_1}	0.0059	0.0059	0.0042
	(0.0007)	(0.0007)	(0.0006)
σ_{ϵ_3}	0.0021	0.0021	0.0006
	(0.0003)	(0.0003)	(0.0002)
σ_{ϵ_6}	7.3e-8	2.2e-8	0.0005
	(0.7372)	(5.1004)	(0.0001)
σ_{ϵ_9}	0.0013	0.0013	0.0005
	(0.0001)	(0.0002)	(0.0001)
Chi-square	48.020	98.839	41.666
P-value	0.0000	0.0000	0.0000
df	5	5	5

Table 3A
Monte-Carlo experiment results for the maximum likelihood parameter estimator of the Vasicek model for 1, 3, 6 and 9-month yields (500 replications)

T=150 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.0500	0.0600	0.0200	0.8000	0.0010	0.0010	0.0010	0.0010
median	0.0524	0.0601	0.0199	0.8003	0.0010	0.0010	0.0010	0.0010
mean	0.0521	0.0625	0.0199	0.8022	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0254	0.0173	0.0012	0.0920	0.0001	0.0001	0.0001	0.0001
cov. rate								
25% cov. rate	0.2740	0.2040	0.2340	0.2540	0.2360	0.2220	0.2420	0.2200
50% cov. rate	0.4840	0.4220	0.4560	0.5460	0.4800	0.4580	0.4820	0.4280
75% cov. rate	0.9320	0.6600	0.7000	0.8280	0.7520	0.7200	0.7420	0.7160
95% cov. rate	1.0000	0.9080	0.9340	0.9860	0.9620	0.9420	0.9320	0.9260

95% cov. rate for Lagrange multiplier test: 0.9380

T=350 observations

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.0500	0.0600	0.0200	0.8000	0.0010	0.0010	0.0010	0.0010
median	0.0510	0.0612	0.0200	0.7981	0.0010	0.0010	0.0010	0.0010
mean	0.0511	0.0611	0.0200	0.7989	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0295	0.0078	0.0008	0.0942	0.0001	0.0001	0.0000	0.0001
cov. rate								
25% cov. rate	0.2220	0.2260	0.2720	0.2300	0.2240	0.2380	0.2500	0.2000
50% cov. rate	0.4760	0.4120	0.4860	0.4660	0.4340	0.4540	0.4860	0.4200
75% cov. rate	0.8120	0.7180	0.7280	0.8040	0.6880	0.7220	0.7540	0.7020
95% cov. rate	1.0000	0.9260	0.9380	0.9980	0.9260	0.9420	0.9520	0.9420

95% cov. rate for Lagrange multiplier test: 0.9260

Table 3B
Monte-Carlo experiment results for the maximum likelihood parameter estimator of the Vasicek model for 1, 3, 6 and 9-month yields (500 replications)

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.1000	0.5000	0.0500	1.0000	0.0010	0.0010	0.0010	0.0010
median	0.1006	0.5000	0.0499	0.9877	0.0010	0.0010	0.0010	0.0010
mean	0.1006	0.5002	0.0499	0.9972	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0167	0.0058	0.0019	0.1713	0.0001	0.0001	0.0000	0.0001
cov. rate								
25% cov. rate	0.2220	0.2400	0.2720	0.2260	0.2240	0.2380	0.2460	0.2120
50% cov. rate	0.4460	0.4680	0.5100	0.4360	0.4360	0.4540	0.4740	0.4180
75% cov. rate	0.7060	0.7500	0.7200	0.7200	0.6860	0.7180	0.7560	0.7040
95% cov. rate	0.9440	0.9360	0.9500	0.9440	0.9340	0.9360	0.9480	0.9360

95% cov. rate for Lagrange multiplier test: 0.9320

T=350 observations

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.1000	0.0100	0.0500	1.0000	0.0010	0.0010	0.0010	0.0010
median	0.1003	0.0105	0.0499	1.0007	0.0010	0.0010	0.0010	0.0010
mean	0.1021	0.0105	0.0500	1.0019	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0734	0.0026	0.0019	0.0397	0.0001	0.0001	0.0000	0.0001
cov. rate								
25% cov. rate	0.6120	0.2060	0.2760	0.2980	0.2180	0.2300	0.2560	0.2180
50% cov. rate	0.8100	0.3960	0.5160	0.5500	0.4420	0.4540	0.4820	0.4240
75% cov. rate	1.0000	0.6600	0.7160	0.7960	0.6860	0.7160	0.7540	0.7020
95% cov. rate	1.0000	0.8740	0.9480	0.9680	0.9320	0.9400	0.9540	0.9400

95% cov. rate for Lagrange multiplier test: 0.8840

Table 4A
Monte-Carlo experiment results for the approximate quasi–maximum likelihood parameter estimator of the CIR model for 1, 3, 6 and 9-month yields (500 replications)

T=150 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.0600	0.3000	0.0750	-0.3000	0.0010	0.0010	0.0010	0.0010
median	0.0560	0.3215	0.0748	-0.3224	0.0010	0.0010	0.0010	0.0010
mean	0.0580	0.3235	0.0748	-0.3207	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0107	0.0595	0.0045	0.0548	0.0001	0.0001	0.0001	0.0001
cov. rate								
25% cov. rate	0.2060	0.2220	0.2400	0.2080	0.2340	0.2400	0.2460	0.2400
50% cov. rate	0.4040	0.4260	0.4420	0.4120	0.4560	0.4780	0.4720	0.4240
75% cov. rate	0.7040	0.7380	0.7280	0.7540	0.7140	0.7500	0.7180	0.6900
95% cov. rate	0.9540	0.9780	0.9400	0.9800	0.9380	0.9480	0.9440	0.9240

95% cov. rate for Lagrange multiplier test: 0.8960

T=350 observations

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.0600	0.3000	0.0750	-0.3000	0.0010	0.0010	0.0010	0.0010
median	0.0577	0.3150	0.0746	-0.3116	0.0010	0.0010	0.0010	0.0010
mean	0.0583	0.3170	0.0748	-0.3153	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0087	0.0480	0.0029	0.0455	0.0001	0.0001	0.0000	0.0000
cov. rate								
25% cov. rate	0.2580	0.2340	0.2500	0.2620	0.2500	0.2580	0.2420	0.2200
50% cov. rate	0.4500	0.4600	0.4620	0.4520	0.4240	0.4680	0.4320	0.4340
75% cov. rate	0.6960	0.7320	0.7200	0.7220	0.6940	0.6960	0.7360	0.7380
95% cov. rate	0.9180	0.9500	0.9440	0.9600	0.9160	0.9420	0.9600	0.9440

95% cov. rate for Lagrange multiplier test: 0.9060

Table 4B
Monte-Carlo experiment results for the approximate quasi–maximum likelihood parameter estimator of the CIR model for 1, 3, 6 and 9-month yields (500 replications)

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.1000	0.5000	0.0500	-1.0000	0.0010	0.0010	0.0010	0.0010
median	0.0999	0.5008	0.0498	-0.9996	0.0010	0.0010	0.0010	0.0010
mean	0.1000	0.5017	0.0498	-1.0012	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0048	0.0282	0.0019	0.0244	0.0001	0.0000	0.0001	0.0001
cov. rate								
25% cov. rate	0.2320	0.2160	0.2060	0.2300	0.2420	0.2380	0.2260	0.2520
50% cov. rate	0.4460	0.4800	0.4400	0.4400	0.4380	0.4820	0.4800	0.4680
75% cov. rate	0.7720	0.7560	0.7180	0.7840	0.7060	0.7640	0.7320	0.7360
95% cov. rate	0.9620	0.9660	0.9460	0.9680	0.9280	0.9600	0.9460	0.9380

95% cov. rate for Lagrange multiplier test: 0.8900

T=350 observations

T=350 observations

	θ	κ	σ	λ	σ_{ϵ_1}	σ_{ϵ_3}	σ_{ϵ_6}	σ_{ϵ_9}
true value	0.1000	0.0100	0.0500	-1.0000	0.0010	0.0010	0.0010	0.0010
median	0.0591	0.0185	0.0501	-1.0069	0.0010	0.0010	0.0010	0.0010
mean	0.0622	0.0188	0.0501	-1.0070	0.0010	0.0010	0.0010	0.0010
std. dev.	0.0188	0.0077	0.0018	0.0041	0.0000	0.0000	0.0001	0.0001
cov. rate								
25% cov. rate	0.0580	0.1280	0.2360	0.0780	0.2300	0.2580	0.2320	0.2160
50% cov. rate	0.1080	0.2600	0.4700	0.1680	0.4360	0.4580	0.4320	0.4360
75% cov. rate	0.1920	0.5220	0.7440	0.3900	0.7280	0.7340	0.7220	0.7360
95% cov. rate	0.4820	0.9180	0.9640	0.8680	0.9420	0.9380	0.9580	0.9500

95% cov. rate for Lagrange multiplier test: 0.9560

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