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Simulation Based Inference in Moving Average Models

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Abstract / Résumé

We examine several simulation-based estimators for the parameters of a moving average process, including the one initially proposed by Gouriéroux, Monfort and Renault (1993) as well as several extensions based on Gallant and Tauchen (1994). The estimators are also compared and related to procedures recently suggested by Galbraith and Zinde-Walsh (1994).

Nous examinons plusieurs estimateurs basés sur les principes des méthodes de moments simulés et l'inférence indirecte pour des modèles de moyenne mobile. Nous étudions une procédure proposée par Gouriéroux, Monfort et Renault (1993) ainsi que des extensions de l'approche proposée par Gallant et Tauchen (1994). Nous faisons également une comparaison avec les procédures de Galbraith et Zinde-Walsh (1994).

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1. Introduction

There are several estimation procedures for moving average models. A large class of estimators are likelihood-based and involve (numerical) optimization of the exact or approximate likelihood function[1]. Another class relies either on the estimation of a sequence of long autoregressions, as in Hannan and Rissanen (1982) and Koreisha and Pukkila (1990) or else involves nonlinear least squares (see, for instance, Fuller (1976) for a discussion). Durbin (1959) suggested fitting an autoregression of finite order to the data and deriving an estimate for the MA parameter[2]. Recently, Galbraith and Zinde-Walsh (1994) proposed a related estimator which involves approximating, in the Hilbert Norm, a MA model by an autoregressive (AR) model. It uses a minimal Hilbert distance criterion to describe explicitly a functional form between the MA parameters and the "auxiliary" AR parameters.

In this paper, we propose several simulated method of moments (SMM) estimators for the parameters of a Gaussian MA(1) process using a finite autoregression. Gouriéroux, Monfort and Renault (1993) provided a first example of such an estimator. The asymptotically equivalent procedure of Gallant and Tauchen (1993) is also considered. Hence we propose several alternatives SMM estimators and conduct a comparative study through Monte Carlo simulations. One advantage of SMM estimators is that, unlike the Durbin and Galbraith/Zinde-Walsh procedures, they do not require an explicit relation between the MA and AR parameters. Another advantage, stressed by Gouriéroux *et al.* (1993) is that simulation based method appear to correct in small samples the bias due the lag truncation in the AR representation. Moreover, we also reconsider the Galbraith/Zinde-Walsh (G/ZW) estimator and suggest an alternative asymptotic least squares (ALS) procedure that improves its large sample properties. In general, the discussion has focused on the performance of estimators near the boundary of the non-invertibility region where procedures like maximum likelihood estimation (MLE) appear to behave poorly in finite samples. Our Monte Carlo evidence shows that the SMM estimators perform better than Durbin and G/ZW's estimators for values of the MA parameter near unity.

[1] See for instance in Box and Jenkins (1976), Godolphin (1977), Osborn (1977), Ansley (1979) and Ansley and Newbold (1980).

[2] This relatively straightforward procedure has attracted considerable attention in the literature; see, for example, McClave (1973) or Mentz (1977) for a discussion of a statistical properties of Durbin's estimator.

The paper is organized as follows. In section 2, we present our notational framework, introduce the different SMM and ALS estimators and discuss their asymptotic distributional properties. In section 3, we report simulation results comparing the finite sample performance of the different estimators. Section 4 concludes.

2. Notation and Proposed Estimators

For convenience we shall devote our attention exclusively to the first order univariate MA model. Extensions to more general models will be discussed in Section 4. The idea to use simulation-based estimators for MA models as suggested in Gouriéroux *et al.* (1993) is further explored in several directions in this section. Three classes of estimators for the MA parameter will be considered. The first is based on a score principle as put forward by Gallant and Tauchen (1993). The second follows Gouriéroux *et al.* and can be characterized as a SMM estimator based on a Wald principle. Finally, we reconsider the G/ZW estimator and replace their OLS procedure by an ALS estimator[3]. We fix notation in section 2.1. Section 2.2 is devoted to the SMM estimators while 2.3 covers the ALS estimators. In section 2.4, we discuss the symptotic distributions of the proposed estimators.

2.1. Notation and definitions

We consider the time series model

$$X_t = u_t - \theta \ u_{t-1}, t = 1, \dots, T, \tag{2.1}$$

where $0 < |\theta| < 1$ and the u_t are independent, identically distributed, normal random variables with $Eu_t = 0, Eu_t^2 = 1$ for all t. The approximating AR(p) model will be denoted as

$$X_t = \beta_1 X_{t-1} + \cdots + \beta_p X_{t-p} + \varepsilon_t, t = 1, \dots, T, \tag{2.2}$$

where ε_t is a white noise. For a sample of size T, let

$$\hat{\beta}^p_{(T)} = \left[\hat{\beta}, \dots, \hat{\beta}_p\right], \tag{2.3}$$

where $\hat{\beta}_i$ are the ordinary least squares estimates of $\beta_i, i = 1, \dots, p$, in the regression of X_t on $X_{t-1}, \dots, X_{t-p}$. The usual central limit theorem arguments yield that $\left(\hat{\beta}_i - \beta_i\right)$ is $O_p\left(T^{-\frac{1}{2}}\right)$ where

[3] See Gouriéroux, Monfort and Trognon (1985) for discussion.

$$\beta_i = -(-\theta)^i \left[\frac{1-\theta^{2(p-i+1)}}{1-\theta^{2(p+1)}}\right], i = 1, \ldots, p^4 \tag{2.4}$$

2.2. SMM estimators of the MA parameter

For a given value of the MA in (2.1), we can obtain H simulated paths

$$\left[\tilde{X}_1^h(\theta), \ldots, \tilde{X}_T^h(\theta)\right], h = 1, \ldots, H, \tag{2.5}$$

based on independent drawings of u_t, $\left[\tilde{u}_1^h, \ldots, \tilde{u}_T^h\right], h = 1, \ldots, H$. For each of these paths, define

$$\tilde{\beta}_{(T)}^{hp}(\theta) = \underset{\beta}{\text{Argmin}}\ \tilde{Q}_{T,h}^{(p)}(\theta), h = 1, \ldots, H, \tag{2.6}$$

where

$$\tilde{Q}_{T,h}^{(p)}(\theta) = \Sigma_{t=p+1}^{T}\left[\tilde{X}_t^h - \beta_1 \tilde{X}_{t-1}^h - \ldots - \beta_p \tilde{X}_{t-p}^h\right]^2, h = 1, \ldots, H. \tag{2.7}$$

In other words, $\tilde{\beta}_{(T)}^{hp}$ minimizes the least squares criterion where the observed values are replaced by the simulated ones. Moreover, we define an estimator for β as follows:

$$\tilde{\beta}_{TH}^{p}(\theta) = \frac{1}{H}\sum_{h=1}^{H} \tilde{\beta}_{(T)}^{hp}(\theta). \tag{2.8}$$

Alternatively, we can obtain a sequence of $N = TH$ simulated values $\tilde{X}_1, \ldots, \tilde{X}_N$ based on N drawings of the white noise process u_t, to which one would apply the following estimation criterion

$$\tilde{Q}_N^{(p)}(\theta) = \Sigma_{t=p+1}^{N}\left[\tilde{X}_t - \beta_1 \tilde{X}_{t-1}, \ldots, \beta_p \tilde{X}_{t-p}\right]^2 \tag{2.9}$$

Once again one can apply the criterion appearing in (2.6) and define $\beta_N^p(\theta) = \underset{\beta}{\text{Argmin}}\ \tilde{Q}_N^{(p)}(\theta)$. We can now define two indirect estimators of θ as follows:

$$\hat{\theta}_{THp}^{SW} = \underset{\theta}{\text{Argmin}}\left(\hat{\beta}_{(T)}^{p} - \hat{\beta}_{TH}^{p}(\theta)\right)' \hat{\Omega}_T \left(\hat{\beta}_{(T)}^{p} - \hat{\beta}_{TH}^{p}(\theta)\right), \tag{2.10}$$

[4] For later use, β_i in (2.4) will be denoted as a function of θ, namely $\beta_i = \beta_i(\theta)$.

where $\hat{\beta}^p_{(T)}$ is defined as in (2.3) and $\tilde{\beta}^p_{TH}$ as in (2.8) while the weighting matrix is obtained as:

$$\hat{\Omega}_T = \left[\left(\frac{\partial \hat{Q}_T^{(p)}}{\partial \beta} \right) \left(\frac{\partial \hat{Q}_T^{(p)}}{\partial \beta} \right)' \right]^{-1}, \tag{2.11}$$

using the sample data to compute

$$\hat{Q}_T^{(p)} = \sum_{t=p+1}^{T} \left(X_t - \hat{\beta}_1 X_{t-1} - \cdots - \hat{\beta}_p X_{t-p} \right)^2. \tag{2.12}$$

Replacing $\tilde{\beta}^p_{TH}(\theta)$ by $\tilde{\beta}^p_N(\theta)$ in section (2.10) yields a second SMM estimator, namely

$$\hat{\theta}^{SW}_{Np} = \underset{\theta}{\text{Argmin}} \left(\hat{\beta}^p_{(T)} - \tilde{\beta}^p_N(\theta) \right)' \hat{\Omega}_T \left(\hat{\beta}^p_{(T)} - \tilde{\beta}^p_N(\theta) \right). \tag{2.13}$$

Both estimators bear the superscript SW indicating that they are simulation-based exploiting a Wald principle. Gouriéroux *et al.* (1993) show that the estimators defined in (2.10) and (2.13) have the same asymptotic distributions given appropriate regularity conditions. They also argue that in this particular application, the efficiency loss resulting from replacing $\hat{\Omega}_T$ by an identity matrix is marginal and therefore suggest to use the latter in practical applications. A third version of the indirect estimator, following Gallant and Tauchen (1993), can be formulated as follows:

$$\hat{\theta}^{SS}_{Np} = \underset{\theta}{\text{Argmin}} \left[\frac{\theta \tilde{Q}_N^{(p)}}{\partial \beta}(\theta) \right]' \hat{\Omega}_T \left[\frac{\theta \tilde{Q}_N^{(p)}}{\partial \beta}(\theta) \right]. \tag{2.14}$$

The advantage of this estimator is that it does not involve estimating β from the simulated data as it relies on the score function, hence the superscript SS. A fourth estimator, denoted $\hat{\theta}^{SS}_{THp}$, can also be defined by combining arguments in (2.10) and (2.14).

2.3. ALS estimators of the MA parameter

The simulation-based estimators discussed in the previous section lead also to a class of estimators not involving simulation yet based on methods of moments principles. For instance, G/ZW proposed an estimator of θ based on $\hat{\beta}_{(T)}$ and the following approximate explicit relation inferred from the Yule-Walker equations:

$$\hat{\theta}_{G/ZW} = \hat{\beta}_1. \tag{2.15}$$

They utilize the approximation (2.4) to evaluate the estimator's asymptotic bias. Obviously, this just identified estimator could be replaced by an ALS estimator exploiting the result in (2.4). Namely consider the estimator defined as

$$\hat{\theta}_p^A = \underset{\theta}{\text{Argmin}} \left(\hat{\beta}_{(T)}^p - \beta^p(\theta)\right)' \hat{\Omega}_T \left(\hat{\beta}_{(T)}^p - \beta^p(\theta)\right), \tag{2.16}$$

where $\beta^p(\theta) = (\beta_1(\theta), \ldots, \beta_p(\theta))'$ is defined in (2.4). Again, we also consider not using $\hat{\Omega}_T$ as a weighting matrix and rather rely on an asymptotically less efficient procedure. Both estimators have a standard asymptotic distribution as described in Gouriéroux, Monfort and Trognon (1985). One can readily see the direct relationship between the $\hat{\theta}_p^A$ and the simulation-based estimators $\hat{\theta}_{THp}^{SW}$ and $\hat{\theta}_{Np}^{SW}$. The former relies on the asymptotic $O_p\left(T^{-\frac{1}{2}}\right)$ expansion typically used in the central limit theorem development of the OLS estimator. The latter two estimators replace the $\beta^p(\theta)$ expression by simulated or averages of simulated estimators. This difference is important to understand the finite sample behavior of the estimators.

2.4. Asymptotic distributions of suggested estimators

In this section we briefly summarize without any formal proof the asymptotic distributions of the estimators discussed in sections 2.2 and 2.3. To simplify the presentation, consider the following notation $\hat{\theta}_j^i(S_T)$, with $i = SW$, SS and A; $j = THp$, Np (for SW and SS) and $j = p$ (for $i = A$) while S_T represents the weighting matrix satisfying

$$S_o = \lim_{T\to\infty} S_T \tag{2.17}$$

Moreover, let

$$b^p(\theta) = \lim_{T\to\infty} \tilde{\beta}_{(T)}^{hp}(\theta), \tag{2.18}$$

where $\tilde{\beta}_{(T)}^{hp}(\theta)$ is defined by (2.6). As (2.18) is for any arbitrary h the index is dropped from the LHS of the equation. Furthermore, let Ω_0 be the information matrix based on the score function considered. In the notation of Gouriéroux, Monfort and Renault (1993), the following special case corresponds to the problem on hand $K_0 = 0$ and $J_0 = I_0 = \Omega_0$. Moreover, it should be emphasized that $T^{\frac{1}{2}}\left(\hat{\beta}_{(T)}^p - \beta^p(\theta_0)\right)$ has Ω_0 as an asymptotic covariance matrix, where $\beta^p(\theta)$ is given by (2.16), $\hat{\beta}^p_{(T)}$ by (2.3) and θ_0 is the true value of the MA parameter. We first examine the ALS estimator, namely

Proposition 3.1: The ALS estimator defined in (2.16) has, under suitable regularity conditions, the following asymptotic distribution:
$T^{\frac{1}{2}}\left(\hat{\theta}^A_p\left(S_T\right)-\theta_0\right)\underset{T\to\infty}{\overset{d}{\to}} N\left(0,\Sigma^A_p\left(S_0\right)\right)$, where

$$\Sigma^A_p\left(S_0\right)=\left[\frac{\partial\beta^p\left(\theta\right)'}{\partial\theta}S_0^{-1}\frac{\partial\beta^p\left(\theta\right)'}{\partial\theta}\right]^{-1}\frac{\partial\beta^p\left(\theta\right)'}{\partial\theta}S_0\Omega_0S_0\frac{\partial\beta^p\left(\theta\right)}{\partial\theta}\left[\frac{\partial\beta^p\left(\theta\right)'}{\partial\theta}S_0^{-1}\frac{\partial\beta^p\left(\theta\right)'}{\partial\theta}\right]^{-1} \tag{2.19}$$

The proof and regularity conditions appear in Gouriéroux *et al.* (1985). From (2.19) we also note that the optimal choice of S_0 is Ω_0^{-1}. We examine the estimator $\hat{\theta}^{SW}_i$ namely

Proposition 3.2: Under suitable regularity conditions,
$T^{\frac{1}{2}}\left(\hat{\theta}^{SW}_i\left(S_T\right)-\theta_0\right)\underset{T\to\infty}{\overset{d}{\to}} N\left(0,\Sigma^{SW}_p\left(S_0\right)\right)$, where $i=TH_P,\ N_P$, and

$$\Sigma^{SW}_p\left(S_0\right)=\left(1+\frac{1}{H}\right)\left[\frac{\partial b^p\left(\theta\right)'}{\partial\theta}S_0^{-1}\frac{\partial b^p\left(\theta\right)}{\partial\theta}\right]^{-1}\frac{\partial b^p\left(\theta\right)'}{\partial\theta}S_0\Omega_0S_0\frac{\partial b^p\left(\theta\right)}{\partial\theta}\left[\frac{\partial b^p\left(\theta\right)'}{\partial\theta}S_0^{-1}\frac{\partial b^p\left(\theta\right)'}{\partial\theta}\right]^{-1} \tag{2.20}$$

The proof and regularity conditions for this result appear in Gouriéroux *et al.* (1993). We notice two differences between the asymptotic variances appearing in (2.19) and (2.20). First, we observe the scaling factor $\left(1+H^{-1}\right)$ because the simulation-based estimators have a source of randomness produced by simulation. It is easy to show, however, that this scaling effect is relatively minor for values of H say greater than 3 and below 10. Second, we also notice that β^p is replaced b^p, since simulation-based methods rely on a numerical mapping between the β vector and θ. A final proposition, stated without proof establishes the relationship between the score-based and Wald SMM estimators.

Proposition 3.3: Under suitable regularity conditions $\hat{\theta}^{SS}_{Np}\left(\Omega_0^{-1}\right)$ is asymptotically equivalent to $\hat{\theta}^{SW}_{Np}\left(\Omega_0^{-1}\right)$. Further, if H is large, $\hat{\theta}^{SS}_{THp}\left(S_T\right)$ has the same limiting distribution as $\hat{\theta}^{SS}_{Np}\left(S_T\right)$, using usual central limit theorem arguments.

3. Finite-sample performance of estimators: Monte Carlo evidence

The discussion in Section 2 yielded estimators which represent two key modifications in comparison to the early Durbin method and the more recently proposed procedure by Galbraith and Zinde-Walsh (1994), namely the SMM and ALS estimators for θ both involve a set of overidentified moment conditions and the SMM estimators rely on simulations of the MA process unlike the previously proposed procedures. The six new estimators θ_j^i of θ, with $i = SW$, SS and A; $j = TH_p$, N_p (for SW and SS) and $j = p$ (for $i = A$) have known asymptotic distributions discussed in the previous section. According to those (first-order) asymptotic results we know that the ALS estimators are more efficient than the SMM estimator. Yet, in small samples it was noted by Gouriéroux *et al.* (1993) that the simulation-based estimator appeared to outperform the asymptotically most efficient ML estimator. We now turn our attention to a Monte Carlo investigation to further explore these small sample properties. All simulations were performed with Gauss Version 3.0 using 1000 replications. All results appear in Talbe 3.1.

We first consider a comparison of $\hat{\theta}_{G/SW}$ and the estimator based on the ALS principle, i.e. $\hat{\theta}_p^A$. Hence we study first the effect of using overidentifying restrictions to estimate θ from the p-dimensional AR parameter vector. We considered cases of $p = 8$ and 12 with sample sizes equal to $T = 50$ and 200. These settings correspond to those examined by Galbraith and Zinde-Walsh and were used for comparison. Table 3.1 reports the Bias as well as the Root Mean Squared Error (RMSE). Four values of θ were considered namely, $\theta = .10, .50, .90, .99$. We continued to use these values for all other experiments as well (this means they differ from the small Monte Carlo investigation in Gouriéroux *et al.* (1993) who consider $T = 250$, and $H = 1$ and $p = 3$)[5].

For low values of θ we find results of the relative performance of the $\hat{\theta}_{G/SW}$, $\hat{\theta}_p^A$ and simulation based estimators as well as the ML estimator that are in line with the asymptotic efficiency rankings. As there is little dependence this is not surprising particularly with $T = 200$. For θ at .5 this ranking is no longer upheld. For instance the ALS generalization of G/ZW with the identity matrix outperforms the original one, but the simulation-based estimators still lag behind.

As θ approaches the noninvertibility region, like for instance $\theta = .99$, we observe how the simulation based estimators outperform the more traditional estimators. For instance the RMSE of $\hat{\theta}_{G/SW}$ is .1296 for $T = 200$ and $p = 8$ while for $\hat{\theta}_{Np}^{SW}$ it is .836 with a 50 % reduction in bias. The score-based estimator does not seem to do as well, however as its RMSE is .1479. Also the bias is greatly improved by using the SW type estimator. Since the score-based simulation estimator performed relatively poorly we considered improving it by increasing H from $H = 1$ to $H = 3$. The results

[5]For the sake of space we only report results with $p = 8$, results with $p = 12$ are available upon request.

in Table 3.1 show however that this had little effect on the performance of the estimator. To summarize, we can say that the simulation based estimators appear to be quite attractive, as they are simple and exhibit desirable small sample properties improving existing methods. Among the simulation based estimators the one based on the Wald principle seems to be the best.

4. Some extensions

We would like to conclude by suggesting some relatively straight forward extensions of the estimators proposed in section 2. First and foremost, it is relatively straightforward to extend the SMM estimators to higher order MA models. This can be done without any major modification. Second, we can also consider multivariate MA models. Such extensions are a bit more involved, but conceptually fairly straightforward.

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Table 3.1 Bias and RMSE of Galbraith and Zinde-Walsh, Simulated Method of Moments and Asymptotic Least Squares Estimators

T	θ	$\hat{\theta}_{G/ZW}$		$\hat{\theta}_{p}^{A}$ *with Identity*		$\hat{\theta}_{p}^{A}$ *with* $\hat{\Omega}$		$\hat{\theta}_{Thp}^{SW}$ *or* $\hat{\theta}_{Np}^{SW}(H=1)$		$\hat{\theta}_{Thp}^{SS}$ *or* $\hat{\theta}_{Np}^{SS}(H=1)$		$\hat{\theta}_{Np}^{SS}(H=3)$	
		RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias
50	.10	.1664	.0029	.1940	.0080	.2375	.0068	.2455	-.0184	.2455	-.0085	.2451	-.0122
200	.10	.0730	-.0004	.0746	.0002	.0845	.0029	.1042	.0002	.1128	-.0039	.1128	-.0024
50	.50	.1666	-.0069	.1525	-.0040	.1656	-.0088	.2000	-.0714	.2379	-.0857	.2408	-.0865
200	.50	.0730	-.0024	.0677	.0002	.0732	.0000	.0874	-.0111	.1001	-.0270	.1041	-.0245
50	.90	.1778	-.0526	.7250	.0055	1.9179	.2092	.2056	-.0389	.2409	-.1814	.2377	-.1718
200	.90	.0854	-.0411	.0750	-.0644	.1266	-.0623	.0899	.0008	.1019	-.0702	.1012	-.0641
50	.99	.2073	-.1157	.3521	-.1160	1.8281	.0107	.1681	-.0442	.2906	-.2472	.2867	-.2379
200	.99	.1296	-.1048	.1384	-.1326	.1299	-.1228	.0836	-.0374	.1488	-.1296	.1481	-.1279

Notes: $\hat{\theta}_{G/ZW}$ denotes the estimator proposed by Galbraith and Zinde-Walsh (1994), $\hat{\theta}_{p}^{A}$ is the ALS estimator as defined in (2.16), $\hat{\theta}_{Np}^{SS}$ *and* $\hat{\theta}_{THp}^{SW}$ refer to the SMM estimators based on the score principle as defined in (2.14). In the same vein, $\hat{\theta}_{Np}^{SW}$ *and* $\hat{\theta}_{THp}^{SW}$ are a notation for the SMM estimators based on the Wald principle as defined in (2.13) and (2.14).