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Tim Bollerslev, Eric Ghysels

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On Periodic Autoregressive Conditional Heteroskedasticity*

Tim Bollerslev† *and Eric Ghysels*‡

Abstract / Résumé

Asset returns exhibit clustering of volatility throughout the year. This paper proposes a class of models featuring periodicity in conditional heteroskedasticity. The periodic structures in GARCH models share many properties with periodic ARMA processes studied by Gladyshev (1961), Tiao and Grupe (1980) and others. We describe the relation between periodic GARCH processes and time-invariant (seasonal) GARCH processes. Besides the periodic GARCH or P-GARCH process, we also discuss P-IGARCH, PI-GARCH, P-ARCH-M and P-EGARCH processes. Extensions to multivariate ARCH processes are studied as well. Moreover, we also consider periodicity in the common persistence of volatility for several series. A quasi-maximum likelihood estimator following Bollerslev and Wooldridge (1992) is defined and a LM test for periodicity derived from it. The models are applied to several asset pricing series.

Dans cette étude, nous proposons une classe de processus ARCH périodiques. Cette structure est semblable à celle des processus linéaires périodiques. Les procésus P-ARCH partagent beaucoup de similarités avec les processus périodiques linéaires mais ont aussi, à cause des non linéarités, des caractéristiques spécifiques. Nous étudions de façon analytique les pertes d'efficacité en terme de prévisions dues à des erreurs de spécifications lorsque les données suivent un processus P-ARCH et qu'un modèle ARCG (saisonnier) est estimé. Le papier inclut également une étude de Monte Carlo qui complémente les résultats théoriques et une appliction au taux de change DM - livre Sterling. Plusieurs extensions, telles que P-EGARCH et P-IGARCH, sont aussi proposées.

Key words: volatility clustering, seasonality, periodic structures, ARCH, GARCH, P-GARCH, exchange rates
Mots clés : persistance dans la volatilité, structures périodiques, taux de change

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† Department of Finance, KGSM, NorthWestern University, and N.B.E.R.

‡ C.R.D.E., Université de Montréal, and CIRANO.

1. Introduction

Several authors have documented seasonal effects in means and standard deviations of monthly stock market returns and dividends. The most recent empirical studies documenting such effects include Schwert (1990), Gallant, Rossi and Tauchen (1992) and Bollerslev and Hodrick (1992). Gallant, Rossi and Tauchen (1992, Table 1) report, for instance, that the variance of the Standard and Poor composite price index in October is almost a tenfold of the variance for, say, March. Moreover, the variance in November is almost twice that of October and hence almost twenty times that of March. Bollerslev and Hodrick found further corroborating evidence regarding seasonalitiy in *conditional* heteroskedasticity for NYSE dividend yields. Indeed, they found significant seasonal lags in ARCH models. In a similar spirit, it is worth noting that Shiller (1992) observed that ten out of the twenty-five stock market crashes which occurred in the U.S. since 1928 were concentrated in one month only, the month of October.[1]

The fact that asset returns exhibit volatility clustering throughout the year is quite interesting both from a theoretical point of view as well as a practical one. Indeed, the intra-year predictability of stock market volatility raises many questions of theoretical interest. For instance, one can think of seasonal habit persistence in preferences and its effect on asset pricing, as documented in Hansen and Sargent (1990) or the fairly regular and institutionalized rhythm of releasing information to the general public, like annual corporate reports and dividend announcements or the calendar of releases of economy-wide economic data by government agencies. Such factors, and many others, contribute to the volatility being structured with month-specific patterns, and many theoretical models could shed light on the dynamic pattern that should emerge. Besides the theoretical questions, another research agenda arises, namely, how to judiciously choose a parametric structure to capture the dynamics of seasonal conditional heteroskedasticity.[2]

It will be helpful to first recall some commonly used time-series models to forecast seasonality in the mean. The framework generally adopted is that of seasonal ARIMA models, possibly involving an unobserved component structure, as discussed, for instance, by Nerlove *et al.* (1979), Bell and Hillmer (1984), Hylleberg (1986),

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[1] Shiller defined a crash as a drop in stock market prices exceeding 6 % between successive trading days. This bears a relation to the periodic stochastic switching-regime models, i.e., models with season-dependent hazard rates of regime switching, discussed in Ghysels (1992, 1993a, b).

[2] This paper proposes a class of models featuring periodicity in conditional heteroskedasticity. An alternative approach, not pursued here, is that of stochastic switching-regime models with a different Markov switching scheme throughout the year which also results in periodic conditional heteroskedasticity as shown in Ghysels (1991, 1992, 1993a, b).

Ghysels (1990), among others. The basic idea of a linear time-invariant autoregressive structure involving seasonal lags can easily be adopted as a possible parameterizaton for the conditional variance. Such would lead to a seasonal ARCH model, as used, for instance, by Bollerslev and Hodrick (1992). An alternative approach to analyzing the mean behavior of seasonal parameter series is to employ ARIMA models whose parameters change seasonally. Initially proposed by Gladyshev (1961), such models have gained considerable interest in recent years. These models, referred to as periodic models because of the seasonal parameter variation, are now well documented both with respect to their theoretical properties as well as their empirical relevance. Tiao and Grupe (1980), for instance, establish the link between the former class of models, namely, seasonal ARIMA models, and periodic ARIMA models. For economic time series, empirical evidence supporting periodic linear structures was documented, for example, by Osborn (1988), Osborn and Smith (1989) for the U.K. and Ghysels and Hall (1992a) for a large class of U.S. macroeconomic seasonally unadjusted data. The periodic parameter variation to capture the repetitive seasonal behavior can be used to construct conditional heteroskedasticity analogues of periodic ARIMA models. In its simplest form, one can consider a periodic ARCH or P-ARCH model. Such model is autoregressive in conditional heteroskedasticity with seasonally varying autoregressive coefficients. This is a first of several models introduced in the paper. One should expect a relationship between P-ARCH models and seasonal GARCH processes whereby (1) P-ARCH models outperform seasonal GARCH processes in terms of volatility predictability and by the same token, (2) a seasonal GARCH representation entails an information loss relative to P-ARCH structures. Both observations emerge as an analogue to the results obtained by Tiao and Grupe (1980) for periodic ARIMA and seasonal ARIMA models. Yet, the analogue between ARCH models and linear structures in the mean only goes through for weak GARCH models, as defined by Drost and Nijman (1993). Indeed, a strong ARCH structure, which is quite often implicitly imposed through ML estimation, does not yield a direct correspondence between a representation with seasonality in the laws and one with seasonality in the lags. For there to be such a correspondence, we first need to weaken the periodic ARCH representation by only considering the linear projections figuring in a weak GARCH model. This means that the information loss alluded to before is more severe with ARCH models than with linear structures.

Section 2 is devoted to P-GARCH models - with P-ARCH as a special case - with a discussion of their stochastic properties and the relationship with seasonal GARCH models. More specifically, a Tiao-Grupe-type formula is introduced for (weak) ARCH structures. In section 3, the notion of periodicity in conditional heteroskedasticity is extended to IGARCH and EGARCH models. Moreover, periodically integrated GARCH or PI-GARCH are also discussed as well as

P-ARCH-M models. This section also covers multivariate periodic ARCH models and periodicity in common persistence. Estimation and hypothesis testing is covered in section 4. Empirical models of periodic stock market volatility appear in section 5.

2. On The Periodic GARCH Model and Weak GARCH

Since the seminal paper by Engle (1982), the autoregressive model of conditional heteroskedasticity and its generalizations are now widely applied.[3] Consider first the ARCH process for ϵ_t, namely:

$$E[\epsilon_t \mid \Omega^{\epsilon}_{t-1}] \quad 0 \tag{2.1a}$$

$$E[\epsilon_t^2 \mid \Omega^{\epsilon}_{t-1}] \equiv \sigma_t^2 = \omega_0 + \sum_{i=1}^{p} \alpha_i \epsilon_{t-1}^2 \tag{2.1b}$$

where Ω^{ϵ}_{t-1} is the usual Borel field filtration based on the realization of the $\{\epsilon_t\}$ process up to t − 1. Now, instead of having a fixed parameter structure, one may draw on the similarity of the AR(p) model and periodic AR processes to consider a time-varying coefficient model for conditional heteroskedasticity in the following manner:

$$E[\epsilon_t \mid \Omega^{\epsilon}_{t-1}] = 0 \tag{2.2a}$$

$$E[\epsilon_t^2 \mid \Omega^{\epsilon}_{t-1}, s] = \sum_{s=1}^{S} d_{st}\omega_{0s} + \sum_{i=1}^{p}\sum_{s=1}^{S} d_{st}\alpha_{is}\epsilon_{t-i}^2 \tag{2.2b}$$

Note first that s appears in the conditioning set in (2.2b). This indicates that s is based on an observable stage of a periodic cycle with length S. The coefficients vary periodically as $d_{st} = 1$ if s is the stage of the periodic cycle at time t and zero otherwise. The most straightforward case is where the periodic cycle is purely repetitive, like $d_{st} = 1$ if $s = t \bmod S$. In some cases though, s may be governed by a variable deterministic cycle with upper bound S. Daily data provide an excellent example. Nontrading days usually take place after every fifth trading day, but some weeks have holidays which interrupt the weekly pattern. In such a case, $S = 5$, but not all trading day cycles attain five consecutive trading days. It should also parenthetically be noted that the lag length p in (2.2b) is independent of S. Throughout the paper, this will be assumed with loss of generality, as p may be set equal to the maximal order of lags

[3] For a recent survey of the empirical literature, see Bollerslev, Chou and Kroner (1992). Theoretical developments, for instance, are surveyed in Bera and Higgins (1992) and Bollerslev, Engle and Nelson (1993).

across all periods. A generalization of (2.1) is the GARCH model, introduced by Bollerslev (1986), which takes the form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{2.3}$$

which can be rewritten as

$$\epsilon_t^2 = \omega_0 + \sum_{i=1}^{Max(p,q)} (\alpha_i + \beta_i)\epsilon_{t-i}^2 + \nu_t - \sum_{j=1}^{p} \beta_j \nu_{t-j} \tag{2.4}$$

where $\nu_t = \epsilon_t^2 - \sigma_t^2$. By definition, ν_t is serially uncorrelated with mean zero. Hence, the representation in (2.4) of the GARCH (p,q) process can be interpreted as an autoregressive moving average process in ϵ_t^2 of orders m = max{p,q) and p respectively. Suitable regularity conditions, as discussed for instance by Bollerslev (1986), ensure that the $\{\epsilon_t^2\}$ process is covariance stationary and hence has a Wold representation as well as spectral decomposition. The analogue of a P-GARCH process defined as

$$\epsilon_t^2 = \sum_{s=1}^{S} d_{st}\omega_{0s} + \sum_{i=1}^{p}\sum_{s=1}^{S} d_{st}(\alpha_{is} + \beta_{is})\epsilon_{t-i}^2 + \nu_t - \sum_{j=1}^{q}\sum_{s=1}^{S} d_{st}\beta_{js}\nu_{t-j} \tag{2.5}$$

where $\nu_t = \epsilon_t^2 - E[\epsilon_t^2 | \Omega_{t-1}^{\epsilon}, s]$ becomes quite apparent. Similar to the periodic ARMA processes, as discussed for example by Tiao and Grupe (1980), which are characterized by a time-varying correlation structure, one can interpret (2.5) as a process with a time-varying but periodic correlation structure in $\{\epsilon_t^2\}$ [see Bollerslev (1988) for an elaborate discussion of the autocorrelation structure of GARCH (p,q) processes].

Yet, the similarities between periodic ARMA and periodic GARCH processes do not carry through straightforwardly. Indeed, the class of GARCH processes is not closed under temporal and cross-sectional aggregation, because the nonlinearities severely complicate both forms of aggregation [see Drost and Nijman (1993)]. It is therefore not possible without further qualifications to apply the formula presented by Tiao and Grupe (1980) to characterize the relationship between periodic GARCH and seasonal fixed parameter GARCH processes. This formula essentially amounts to averaging out the autocorrelation structure across all seasons. Osborn (1991) notes that in some cases, one can draw a direct comparison between the operation of averaging out correlations and that of cross-sectional aggregation. Consequently, we need to weaken the periodic GARCH structure in (2.5) to avoid the complications of nonlinearities before applying Tiao and Grupe's formula. To facilitate the discussion, let us first rewrite equation (2.5) as:

$$\sigma_t^2 = \omega_{0s} + A_s(L)\epsilon_{t-i}^2 + \beta_s(L)\sigma_{t-1}^2 \tag{2.6}$$

where $A_s(L) = \sum_{i=1}^{p} \alpha_{is} L^i$, $\beta_s(L) = \sum_{j=1}^{q} \beta_{js} L^j$ and s = t mod S. Following Drost and Nijman (1993), we consider a weak P-GARCH process when σ_t^2 in (2.6) corresponds to the best linear projection of ϵ_t^2on the space spanned by $\{1, \epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_{t-1}^2, \epsilon_{t-2}^2, ...\}$ *given* period s. More specifically,

$$E[\epsilon_t^2 - \sigma_t^2 \mid s] = E[(\epsilon_t^2 - \sigma_t^2)\, \epsilon_{t-i} \mid s] = E[(\epsilon_t^2 - \sigma_t^2)\epsilon_{t-i}^2 \mid s] = 0 \tag{2.7}$$

yielding the following alternative representation for (2.6):

$$\epsilon_t^2 = \omega_{0s} + (A_s(L) + B_s(L))\epsilon_{t-1}^2 + B_s(L)\nu_{t-1} + \nu_t \tag{2.8}$$

where s = t mod S and $\nu_t = \epsilon_t^2 - P(\epsilon_t^2 \mid \epsilon_{t-1}, ..., \epsilon_{t-1}^2, ..., s)$ with $P(\cdot)$ the linear projection.[4]

Note that the projections in (2.7) still involve seasonal conditioning and therefore produce a periodic autocorrelation structure. The specification of weak P-GARCH obviously entails an information loss, since the conventional P-GARCH process defines σ_t^2as the conditional expectation of ϵ_t^2 based on the full information set implied by the Borel σ-field filtration of $\{\epsilon_t\}$ augmented with seasonal conditioning. The Tiao-Grupe formula amounts to the removal of the seasonal conditioning in (2.7) and results in a process which is a member of the class of weak GARCH processes introduced by Drost and Nijman (1993). Once we restrict ourselves to the class of weak GARCH processes, we can carry out the mechanics of the Tiao-Grupe formula. Such operation begins with constructing a skip-sampled vector representation of the squared residuals collecting all observations over a single periodic cycle. Since there are S such squared residuals, let us define $\underline{\epsilon}_t^2 \equiv (\epsilon_{S\tau}^2, ..., \epsilon_{S(\tau-1)+2}^2, \epsilon_{S(\tau-1)+1}^2)$ where τ is an "annual" time index.[5] Likewise, we can define $\underline{\nu}_\tau$ as a S × 1 vector over an entire periodic cycle of innovations appearing in the weak P-GARCH (p,q) model (2.8). Since the vectors obtained this way cover an entire periodic cycle, they encompare all possible parameter variations and therefore yield a time invariant vector system. It may be useful to consider a simple case where S = 2 with alternating periods which

[4] It is worth parenthetically noting at this point that in many practical applications one may restrict the $B_s(L)$ polynomial independent of s, resulting in a P-GARCH process with periodic patterns only in the AR part.

[5] We use the term annual here in analogy with common applications of periodic models to quarterly or monthly data. Yet, if we model ARCH processes for daily or intraday sampling frequences, then τ may correspond to a weekly time scale with a vector representation of daily series or even to daily sampling of a vector of hourly processes, etc.

each have a GARCH (1,1) structure. This illustrative example will also be used later for numerical computations. Then one can construct, using equation (2.7), the following bivariate representation:

$$\begin{pmatrix} 1 & -(\alpha_{12}+\beta_{12}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon^2_{\tau,2} \\ \epsilon^2_{\tau,1} \end{pmatrix} = \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (\alpha_{11}+\beta_{11}) & 0 \end{pmatrix} \begin{pmatrix} \epsilon^2_{\tau-1,2} \\ \epsilon^2_{\tau-1,1} \end{pmatrix} + \begin{pmatrix} \beta_{12} & 0 \\ 0 & \beta_{11} \end{pmatrix} \begin{pmatrix} \nu_{\tau,2} \\ \nu_{\tau,1} \end{pmatrix} \tag{2.9}$$

which can also be written as:

$$\underline{\epsilon}^2_\tau = \begin{pmatrix} \omega_2 + (\alpha_{12}+\beta_{12})\,\omega_1 \\ \omega_1 \end{pmatrix} + \begin{pmatrix} \alpha_{11}+\beta_{11})(\alpha_{12}+\beta_{12}) & 0 \\ \alpha_{11}+\beta_{11} & 0 \end{pmatrix} \underline{\epsilon}^2_{t-1} + \begin{pmatrix} \beta_{11} & \beta_{11}(\alpha_{12}+\beta_{12}) \\ 0 & \beta_{11} \end{pmatrix} \underline{\nu}_\tau \tag{2.10}$$

Hence, we obtain a bivariate time invariant representation of the $\underline{\epsilon}^2_\tau$process.

Under suitable regularity conditions regarding the largest eigenvalue of det(I − Aλ), where A is the first-order lag matrix of coefficients in (2.10), there is a Wold decomposition and spectral representation of the $\underline{\epsilon}^2_\tau$ process. This analysis is not constrained, of course, to P-GARCH(1,1) processes. For any weak P-GARCH(p,q) process, as specified in (2.8), we can derive a fundamental MA representation:

$$\underline{\epsilon}^2_\tau = \omega_0 + \sum_{j=0}^{+\infty} \bar{A}_j \, \underline{\nu}_{t-j}. \tag{2.11}$$

The elements of $\bar{A}_j$ are determined by the polynomials of the period GARCH model in (2.8), similar to (2.10) and its resulting MA representation. From (2.11), we can obtain a multivariate covariance generating function and spectral representation:

$$F(e^{-i\omega}) = \bar{A}(e^{-i\omega})\, Q\, \bar{A}(e^{i\omega})' \quad -\pi \leq \omega \geq \pi \tag{2.12}$$

where Q is the covariance matrix of the $\{\underline{\nu}_\tau\}$.[6]

[6] An important distinction has to be made here between the usual multivariate ARCH processes, as studied in various forms by Baba, Engle, Kraft and Kroner (1990), Bollerslev, Engle and Wooldridge (1988), Diebold and Nerlove (1989), among others. Indeed, unlike the usual multivariate ARCH process, the vector of ARCH processes in (2.8) does not involve any conditional cross-covariances as components of the vector. This is a consequence of the fact that each component of the vector represents the same process sampled at a different time.

Following Tiao and Grupe (1980), one can establish that

$$\sigma_v^2 \left| \frac{\sum_{i=1}^{Max(P,Q)} (\alpha_i + \beta_i) e^{-i\omega}}{1 - \sum_{j=1}^{Q} \beta_j e^{-i\omega}} \right|^2 = R(e^{-i\omega})\ F(e^{-Si\omega})\ R(e^{-i\omega})' \tag{2.13}$$

where $R(e^{-i\omega}) = \sqrt{S}\ (1\ e^{-i\omega}\ \ldots\ e^{-(S-1)i\omega})$. Equation (2.13) establishes a computable formula linking the parameters α_{is} and β_{js} for $s = 1, \ldots, S$ i = 1, ..., p, j = 1, ..., q and a weak GARCH (P,Q) process with parameters α_i, β_j, for i = 1, ..., P and j = 1, ..., Q. It also implicitly establishes a relationship between p, q and P, Q, i.e., the order of the GARCH process. In general, this relationship is not trivial, as discussed in detail by Osborn (1991), yet one knows that $P \geq S$ whenever $p \neq 0$. Unfortunately, the Tiao-Grupe formula does not yield an analytical characterization of the correspondence between the weak GARCH(P,Q) parameters and the S parameter vectors of the weak P-GARCH(p,q). Therefore, we have to rely on numerical computations.

The numerical tool provided by Tiao-Grupe's formula can be put to use to evaluate the loss of prediction accuracy foregone from ignoring seasonal conditioning in the information sets used to formulate σ_t^2. We will focus exclusively on the weak P-GARCH model and its relationship with weak GARCH. Hence, the information loss associated with the relaxation from strong P-GARCH to weak P-GARCH will not be assessed here. Obviously, evaluating the prediction accuracy of ARCH models is not very straightforward, as the prediction error distribution is generally complex, involves higher moments and is leptokurtic. Despite its drawbacks, we will use the minimum MSE prediction criterion to assess the information loss attributable to foregoing periodicity in the stochastic structure. Although, it is worth stressing the limitations of the MSE criterion, as it puts equal weight to forecast errors in a heteroskedastic environment. Following Kolmogorov (1941) and Janacek (1975), one can compute the minimum MSE from a linear predictor for a process with known spectral density $f(\omega)$. Namely,

$$\text{MSE} = \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log 2\pi f(\omega) d\omega\right]. \tag{2.14}$$

This result can be directly applied to equation (2.13), yielding the MSE of the weak GARCH representation of the periodic process:

$$\mathrm{MSE}_{WG} \equiv \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log 2\pi\,(R(e^{-i\omega})\ F(e^{Si\omega})\ R(e^{-i\omega})')\ d\omega\right] \quad (2.15)$$

where the WG index refers to the weak GARCH representation. The MSE for the weak P-GARCH process on the other hand is the sum of the MSE's for each season separately. Therefore,

$$\mathrm{MSE}_{WPG} \equiv \sum_{i=1}^{S} \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log 2\pi\,(F_{ii}(e^{-i\omega}))\ d\omega\right] \quad (2.16)$$

where $F_{ii}(\cdot)$ are the diagonal elements of the spectral density matrix defined in (2.12).

[Numerical computations to be included.]

3. Conditional Heteroskedasticity Models and Periodicity

The idea to use periodic structures in formulating models of conditional heteroskedasticity is not only limited to the GARCH process. It can be exploited in other models, both univariate and multivariate. Here, we will introduce several extensions of the basic structure developed in section 2, hoping that they may lead us to a better understanding and/or prediction of the observed volatility clustering. It would be easy to simply take all classes of processes suggested so far and define a periodic version for each of them. Hence, EGARCH would lead to P-EGARCH, IGARCH to P-IGARCH, TARCH to P-TARCH, STARCH to P-STARCH and so on. Obviously, such an unguided generalization is not very useful. Instead, let us focus on a few cases which might be the most interesting to consider.

Let us first return to equation (2.8) which represented the general (weak) P-GARCH process. In many applications where the conventional GARCH model is fitted to high frequency data, one finds the parameter estimates of the AR and MA polynomials sum to approximately one. This has led to the so-called Integrated GARCH(p,q) or IGARCH(p,q) model proposed by Engle and Bollerslev (1986). For periodic heteroskedasticity, it may be useful to extend this to the P-IGARCH process, based on equation (2.8) with the restriction that:

$$\alpha_s(1) + \beta_s(1) = 1 \quad \forall s = 1, ..., S. \quad (3.1)$$

This restriction is the ARCH analogue of the I(1) restriction in linear periodic ARMA processes considered by Ghysels and Hall (1993a) who developed tests for the null hypothesis of an integrated process. A special case of (3.1) is where $\beta_s(1) = \beta(1)\ \forall s$.

This restricts the periodic pattern to the AR part, which as noted in the previous section is of practical use.

It is well known that GARCH processes capture the thick tailed distribution of stock market returns and exchange rate data and are able to mimic the observed volatility clustering. Yet, they are not well suited for capturing leverage effects, i.e., asymmetric responses in the conditional variance function. The exponential GARCH model proposed by Nelson (1991) has $\{\ln(\sigma_t^2)\}$ follow an ARMA process. In addition, the innovation process is constructed such that positive and negative shocks have a distinct effect. The unconstrained P-EGARCH model would then be written as

$$\ln(\sigma_t^2) = \omega_s + \frac{A_s(L)}{B_s(L)} (\theta_s z_{t-1} + \gamma_s[|z_{t-1}| - E|z_{t-1}|]). \tag{3.2}$$

Obviously, the process (3.2) as such is easily overparameterized. Suppose we restrict ourselves to the periodicity due to nontrading day effects, i.e., $S = 2$, but with a variable though perfectly predictable periodic pattern. Then for a P-EGARCH (1,1) model, one has eight parameters to fit. For higher order and for a more complicated periodic cycle, this rapidly increases at a rate S [(p+q) + 3]. The case where $A_s(L) = A(L)$ and $B_s(L) = B(L)$ $\forall s = 1, ..., S$ corresponds to processes with periodic asymmetries. Hence, a negative shock after, say a nontrading day, has a different impact than on any other day. Conversely, with $\theta_s = \theta$, $\gamma_s = \gamma$ $\forall s = 1, ..., S$, and periodic polynomials, then dynamic responses differ, similar to the P-GARCH model studied in the previous section.

Nelson (1990) shows that the EGARCH model approximates in discrete time a diffusion of the type:

$$d[\ln(y_t)] = \theta\sigma_t^2\, dt + \sigma_t\, dW_{1t} \tag{3.3a}$$

$$d[\ln(\sigma_t^2)] = -\alpha[\ln(\sigma_t^2) - \beta]\, dt + dW_{2t}, \tag{3.3b}$$

where W_{1t} and W_{2t} are independent standard Wiener processes. This result prompts the question whether there is a diffusion limit to a P-EGARCH process. Ghysels and Jasiak (1993) introduced stochastic volatility models with time deformation, i.e., the Ornstein-Uhlenbeck process (3.3b) evolves in an operational time r which differs from calendar time t. There is a functional relationship $r = g(t)$ between the operational and calendar time scales. The changes in operational time, denoted $\Delta g(t)$, between two consecutive discrete sample points $t - 1$ and t are assumed to be measurable with respect to the usual time filtration σ-algebra in calendar time. Ghysels and Jasiak (1993) adapt a logistic function suggested by Stock (1998):

$$\Delta g(t) = \exp(c' z_{t-1}) \, / \sum_{t=1}^{T} \exp(c' z_{t-1}) \tag{3.4}$$

where z_{t-1} is a vector containing variables known at t − 1. Candidate processes which set the pace of the operational clock suggested by Ghysels and Jasiak include past volume and price changes and their absolute value. Such parameters relate to the flow of information arriving on the floor of the stock market. The stochastic volatility model (3.3) with the second equation replaced by

$$d[\ln \sigma_r^2] = -\alpha[\ln(\sigma_r^2) - \beta]\, dr + dW_{2r} \tag{3.5}$$

where $\Delta g(t)$ is determined by (3.4) has a discrete time representation

$$\log[(\log y_t - \log y_{t-1})]^2 = \log \sigma_t^2 + \zeta_t \tag{3.6a}$$

$$\log \sigma_t^2 = e^{A\Delta g(t)} \log \sigma_{t-1}^2 + \epsilon_t \tag{3.6b}$$

with $E\zeta_t = -1.27$, $E(\zeta_t + 1.27)^2 = \pi 2/2$, $E\epsilon_t = 0$ and $E\zeta_t^2 = f(\Delta g(t))$.[7] If the time deformation is taken to be purely deterministic, i.e.,

$$\Delta g(t) = \exp\left(\sum_{s=1}^{S} d_{st} c_s\right) \Big/ \frac{1}{T} \sum_{t=1}^{T} \exp\left(\sum_{s=1}^{S} d_{st} c_s\right),$$

then the stochastic volatility model (denoted SVM) appearing in (3.6) becomes a periodic − SVM:

$$\log \sigma_t^2 = \alpha_s \log \sigma_{t-1}^2 + \epsilon_{st} \tag{3.7}$$

the latter replacing (3.6b). Note that both the AR coefficient and the innovation variance take on different values each period. The P-SVM and P-EGARCH models both represent processes with a continuous time SVM with periodic time deformation.[8]

We conclude this section with a brief discussion of multivariate ARCH models. Obviously, conditional ARCH models lead to multivariate extensions which are easily overparameterized. Periodic multivariate extensions surely will not make it easier, on the contrary. There is, however, one direction of extension that may be feasible and useful at the same time. Bollerslev and Engle (1993) consider a multivariate ARCH process:

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[7] See Ghysels and Jasiak (1993) for details.

[8] See Anderson (1993) for a discussion of ARCH-type versus SVM characterizations.

$$\epsilon_t = \Omega_t^{1/2} Z_t \tag{3.8}$$

where ϵ_t is a m × 1 vector process, Z_t is i.i.d. with $EZ_t = 0$ and $EZ_t Z_1' = I$, while Ω_t is a matrix conditional covariance function measurable with respect to the usual t − 1 filtration. They define the process (3.8) to be copersistent in variance if at least one element of $E_0\Omega_k$ is nonconvergent a.s. for increasing k, yet there exists a nontrivial linear combination $\gamma' \epsilon_t$ with finite limit $\lim_{k\to\infty} E_0(\gamma' \Omega_k \gamma) < \infty$. This notion can easily be extended to periodic copersistence. Such a process would satisfy the condition that:

$$\lim_{k\to\infty} E_0 \gamma_s' \Omega_{s+Sk} \gamma_s < \infty \qquad \forall s = 1, \ldots, S \tag{3.9}$$

and independent of the initial time 0. The definition of periodic copersistence amounts to saying, of course, that the stacked process $\underline{\epsilon}_\tau$, containing all periods, in analogy with the discussion in section 2, has a nontrivial linear combination $\underline{\gamma} \equiv (\gamma_1', \ldots, \gamma_S')'$ which is copersistent, as defined by Bollerslev and Engle.

4. Estimation and Hypothesis Testing

A variety of estimation procedures have been suggested for ARCH models. Estimation, hypothesis testing and model selection of ARCH models for univariate as well as multivariate processes are very well covered in the survey articles by Bera and Higgins (1992), and Bollerslev, Engle and Nelson (1993). It is an area of active ongoing research with still many unresolved issues as both articles clearly emphasize. The scope of this section is not to contribute the basic theory of estimation and hypothesis testing as such. Instead, our aim is to discuss issues pertaining to the estimation and testing of periodic ARCH processes. In particular, we propose a LM test for periodicity in ARCH processes.

Suppose we would like to test the null of an aperiodic or conventional ARCH structure against a periodic alternative. A test strategy which is particularly suitable for this purpose is that of the Lagrangian Multiplier test, which was adopted for linear structures by Ghysels and Hall (1992). It is convenient since the periodic alternative involves many more parameters relative to the nonperiodic null, which is easy to estimate, as it corresponds to the usual GARCH, EGARCH, etc. specifications. By a standard prediction error decomposition argument, let the log likelihood function for the periodic model equal the sum of the conditional log likelihoods:

$$L_T(y_T, ..., y_1; (\psi_1', ..., \psi_S')') = \sum_{t=1}^{T} d_{st} \ell_{st}(y_t; \psi_s) \tag{4.1}$$

where $\underline{\psi}' \equiv (\psi_1', ..., \psi_S')$ is the parameter vector of the periodic specification and ℓ_{st} is the log likelihood pertaining to periods s involving ψ_s. It should parenthetically be noted that the vector may consist of $\psi_s' \equiv (\theta_s', \eta_s')$ where η_s is a vector of nuisance parameters used, for instance, to determine the distribution of the conditional heteroskedasticity process innovations. Of course, η_s may be absent when, for example, the distribution is N(0,1) or η may be independent of s. If the conditional density in (4.1) is correctly specified, then under appropriate regularity conditions a central limit theorem argument yields that:

$$T^{1/2}(\hat{\underline{\psi}}_T - \underline{\psi}_0) \to N(0, A_0^{-1}) \tag{4.2}$$

where $\to$ denotes convergence in distribution and A_0 is the information matrix.[9] Let us denote $S_{st}(y_t, \psi_s)$ as the score function of the t^{th} observation which belongs to season s. Then the null hypothesis of interest is:

$$H_0 : \psi_s = \psi_0 \quad \forall s, s = 1, ..., S \tag{4.3}$$

against the alternative that for at least one s the equality in (4.3) does not hold. The scores, when evaluated under the null (4.3) and ψ_0 being the true parameter vector, will satisfy a martingale central limit theorem, and therefore yielding a LM test with standard asymptotic distribution:

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[9] Many aspects regarding MLE of the general class of ARCH processes still remain unresolved. See, e.g., Bollerslev, Engle and Nelson (1993) for further discussion.

$$LM_T^P = T^{-1} \sum_{s=1}^{S} [S_s(\hat{\psi})' \hat{V}^{1} S_s(\hat{\psi})] \overset{d}{\to} \chi^2(d) \tag{4.4}$$

where:

$$S_s(\hat{\psi}) = \sum_{t=1}^{T} d_{st} \, S_{st}(y_t, \hat{\psi}) . \tag{4.5}$$

$\hat{\psi}$ is the MLE under the null (4.3), $\hat{V}$ denotes a consistent estimator of the square submatrix $[A_0]_{1:1}$ with $l = \dim(\psi)$. Finally, the degrees of freedom equal $d = (S - 1)l$. The score test in (4.4) is easy to implement as it only involves estimating the nonperiodic specification and then checking whether the score function evaluated with data from each period s separately is still close to zero. The score test in (4.4) can also be robustified to fit the context of QMLE, as shown by Wooldridge (1990), Bollerslev and Wooldridge (1992). To accommodate the possibility of a misspecified likelihood function, one must modify the matrix V in (4.4), since the outerproduct of gradients and Hessian do not cancel out in the QMLE context. For further details, see, for instance, Bollerslev and Woodridge (1992). Obviously, one can also formulate a Wald and/or LR-type test for the null hypothesis (4.3). Both would involve estimating the unconstrained, i.e., periodic, ARCH process. Here, also, one can allow for the possibility of a QMLE framework. Bollerslev, Engle and Nelson (1993) provide details of such tests.

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