**CAHIER 03-2001**

# WHAT IS A COMMODITY? TWO AXIOMATIC ANSWERS

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February 2001

The author is grateful to S. Barberà, W. Bossert, N. Meddahi, E. Renault, and O. Sprumont for stimulating conversations. He thanks the Social Sciences and Humanities Research Council of Canada for support.

## RÉSUMÉ

Nous examinons sous quelles conditions des préférences à propos d'ensembles d'options de consommation peuvent se réduire à des préférences à propos de paniers de "biens". Nous distinguons les paniers ordinaux, dont les composantes sont définies à une transformation croissante près, des paniers cardinaux, définis à une transformation linéaire positive près.



Mots clés : bien, préférence, mesure ordinale, mesure cardinale



## ABSTRACT

We identify conditions under which preferences over sets of consumption opportunities can be reduced to preferences over bundles of "commodities". We distinguish ordinal bundles, whose coordinates are defined up to monotone transformations, from cardinal bundles, whose coordinates are defined up to positive linear transformations only.



Key words : commodity, preference, ordinal measure, cardinal measure

## 1. Introduction

This note is an attempt to define the notion of commodity. We start from the intuition that a commodity is "something all agents care about", which suggests a subjective (i.e., preference-based) and collective definition. We take the view that agents have preferences over subsets of a world of consumption opportunities and we propose to define a commodity as a subset of the world to which the preferences of the agents confer a separate status.

In order to be labeled as a commodity, a subset of the world ought to possess, at least, the following two properties. First, all agents should have identical preferences over its subsets. This "agreement" property allows us to measure the commodity in an ordinal sense: a subset of that commodity contains more of it than another if all agents prefer the former over the latter. Second, an agent who prefers a subset of a commodity to another subset of the same commodity should also prefer the union of the former with any subset of the world disjoint from that commodity over the union of the latter with that disjoint subset. This "separability" condition guarantees that our ordinal measure is meaningful: whether a subset of the world contains more of a given commodity than another does not depend on whether the former contains more or less of the *other* commodities than the latter. Under these two restrictions, the preferences of the agents can be reduced to orderings over "ordinal bundles of commodities", i.e., vectors whose coordinates are defined up to arbitrary monotone transformations: see Proposition 1 for a formal statement.

This is not, however, the traditional view of a commodity, as expressed in Debreu (1959), chapter 2, for instance. Reducing the preferences to orderings over "cardinal bundles of commodities", whose coordinates are defined up to positive linear transformations, requires to strengthen the agreement property. One possibility is to demand that all partitions of every subset of a given commodity be Pareto efficient, so that agents never find it profitable to exchange subsets of the same commodity against each other. Proposition 2 states that this "no profitable exchange" requirement forces the form of homogeneity proper to cardinal commodities.

## 2. Two definitions

Let $W$ be a nonempty set, which we call a *world,* and let $\Omega$ be a $\sigma$-algebra of subsets of $W$. Let $I$ be a finite set of agents. Each agent $i \in I$ is endowed with a preference relation $R_i$ over $\Omega$; strict preference is denoted by $P_i$ and indifference by $I_i$. The following assumptions on the *preference profile* $R = (R_i)_{i \in I}$ are maintained throughout: for every $i \in I$, (i) (monotonicity) for all $A, B \in \Omega$ such that $A \supset B$, $AR_iB$; (ii) (representability) $R_i$ possesses a numerical representation[1]; (iii) (perfect

[1] A numerical representation of $R_i$ is a mapping $u_i : \Omega \to \mathbb{R}$ such that, for all $A, B \in \Omega$, $AR_iB$ if and only if $u_i(A) \geq u_i(B)$.

divisibility) for every $A \in \Omega$ such that $AP_i\emptyset$, there exists $B \subset A$ such that $AP_iBP_i\emptyset$; and (iv) (monotone continuity) for every $A \in \Omega$ and every sequence $\{B_n\}$ in $\Omega$ such that $B_n \supset B_{n+1}$ and $B_nR_iA$ for all $n$, $\cap_{n=1}^{\infty} B_nR_iA$. Assumptions (i) and (ii) should be familiar to economists and (iii) and (iv) are borrowed from the literature on representing a "probability" relation on a $\sigma$-algebra of events by a measure. A short discussion of these assumptions will be provided at the end of this section.

Let $\mathcal{W}$ be a partition of the world $W$ into finitely many nonempty sets $W^1, ..., W^G \in \Omega$, and write $\Omega^g := \{A \cap W^g : A \in \Omega\}$ for every $g = 1, ..., G$. We call such a partition a *system of ordinal commodities* (*for the preference profile* $R$) if there exist monotone[2] mappings $x^g : \Omega^g \to \mathbb{R}_+$ (for $g = 1, ..., G$) and monotone[3] preference relations $\succcurlyeq_i$ over $\mathbb{R}_+^G$ (for $i \in I$) such that, for all $A, B \in \Omega$ and $i \in I$,

$$AR_iB \Leftrightarrow (x^1(A \cap W^1), ..., x^G(A \cap W^G)) \succcurlyeq_i (x^1(B \cap W^1), ..., x^G(B \cap W^G)).$$

If the mappings $x^1, ..., x^G$ can be chosen to be countably additive measures, we call $\mathcal{W}$ a *system of cardinal commodities.*

If $\mathcal{W}$ is a system of ordinal commodities for $R$, it is easily seen that $R$ satisfies the following two conditions with respect to $\mathcal{W}$ :

*Agreement.* For all $i, j \in I$ , all $g = 1, ..., G$, and all $A, B \in \Omega^g$, $AR_iB \Leftrightarrow AR_jB$.

*Separability.* For all $i \in I$, all $g = 1, ..., G$, all $A, B \in \Omega^g$, and all $C \in \Omega \backslash \Omega^g$, $AR_iB \Leftrightarrow (A \cup C)R_i(B \cup C)$.

The converse statement is almost as obvious. Formally,

**Proposition 1.** *A finite partition $\mathcal{W}$ of the world $W$ is a system of ordinal commodities for the preference profile $R$ if and only if $R$ satisfies Agreement and Separability with respect to $\mathcal{W}$.*

**Proof.** We omit the straightforward verification of the "only if" statement. To prove the "if" statement, fix a preference profile $R$ satisfying assumptions (i) to (iv) and a finite partition $\mathcal{W} = \{W^1, ..., W^G\}$ with respect to which $R$ satisfies Agreement and Separability. Let $u_1$ be a numerical representation of $R_1$; the existence of such a representation is postulated in (ii) and there is no loss in assuming, as we shall, that the range of $u_1$ is contained in $\mathbb{R}_+$. For every $g \in \{1, ..., G\}$, let $u_1^g$ denote the restriction of $u_1$ to $\Omega^g$ : it is monotone because of (i) and we know by Agreement that

$$\text{for all } i \in I \text{ and all } A, B \in \Omega^g, \;\; AR_iB \Leftrightarrow u_1^g(A) \geq u_1^g(B).$$

[2]Monotonicity is with respect to inclusion: $x^g$ is monotone if, for all $A, B \in \Omega^g$, $x^g(A) \leq x^g(B)$ whenever $A \subset B$.

[3]Monotonicity is with respect to the standard partial ordering of $\mathbb{R}_+^G$, and it is understood in the strict sense: using $\succ_i$ for strict preference, the relation $\succcurlyeq_i$ on $\mathbb{R}_+^G$ is monotone if $(a^1, ..., a^G) \succ_i (b^1, ..., b^G)$ whenever $a^g \geq b^g$ for all $g \in \{1, ..., G\}$ and $a^g > b^g$ for some $g \in \{1, ..., G\}$.

Let $X^g = u_1^g(\Omega^g) = \{u_1(A) : A \in \Omega^g\}$ and define $X = \times_{g=1}^G X^g$ : this is a subset of $\mathbb{R}_+^G$.

For the rest of the proof, fix $i \in I$. The crucial observation is that, for any $A, B \in \Omega$,

$$(u_1(A \cap W^1), ..., u_1(A \cap W^G)) = (u_1(B \cap W^1), ..., u_1(B \cap W^G)) \Rightarrow AI_iB. \quad (1)$$

To check this, assume the left-hand side. By Agreement, $(A \cap W^g)I_i(B \cap W^g)$ for every $g = 1, ..., G$. Then, by repeated application of Separability, $A = \cup_{g=1}^G(A \cap W^g)$ $I_i$ $[(B \cap W^1) \cup (A \cap W^2) \cup ... \cup (A \cap W^G)]$ $I_i$ $[(B \cap W^1) \cup (B \cap W^2) \cup ... \cup (A \cap W^G)]$ $I_i ... I_i$ $\cup_{g=1}^G(B \cap W^g) = B$, as desired.

We now construct a relation $\succcurlyeq_i$ over $\mathbb{R}_+^G$ as follows. For all $a = (a^1, ..., a^G), b = (b^1, ..., b^G) \in X$, there exist sets $A, B \in \Omega$ such that $a = (u_1(A \cap W^1), ..., u_1(A \cap W^G))$ and $b = (u_1(B \cap W^1), ..., u_1(B \cap W^G))$. Pick *any* such sets and let

$$a \succcurlyeq_i b \Leftrightarrow AR_iB. \quad (2)$$

Thanks to (1), $\succcurlyeq_i$ is a well-defined relation over $X$; we extend it arbitrarily to $\mathbb{R}_+^G$. It follows directly from (2) that, for all $A, B \in \Omega$,

$$AR_iB \Leftrightarrow (u_1(A \cap W^1), ..., u_1(A \cap W^G)) \succcurlyeq_i (u_1(B \cap W^1), ..., u_1(B \cap W^G)),$$

which completes the proof. ■

It should be clear that different partitions of the world may qualify as systems of ordinal commodities for the same profile. Clearly, minimal systems are of particular interest. Note that the trivial partition $\mathcal{W} = \{W\}$ is always the unique such system if $|I| = 1$. This is an extreme but logical consequence of our view of commodities. According to (both of) our definitions, commodities are determined "subjectively" and "collectively". In particular, the set of commodities changes with the preferences of the members of society and can only grow with the size and diversity of a population. *Per se*, however, commodities need not possess any physical or otherwise objective form of homogeneity.

We now turn to cardinal commodities. If $\mathcal{W}$ is a system of cardinal commodities for the profile $R$, that profile must possess the following additional property:

*Internal Separability.* For all $i \in I$, all $g = 1, ..., G$, and all $A, B, C \in \Omega^g$ such that $C \cap (A \cup B) = \emptyset$, $AR_iB \Leftrightarrow (A \cup C)R_i(B \cup C)$.

**Lemma.** *A finite partition $\mathcal{W}$ of the world $W$ is a system of cardinal commodities for the preference profile $R$ if and only if $R$ satisfies Agreement, Separability, and Internal Separability with respect to $\mathcal{W}$.*

**Proof.** Again, we omit the straightforward verification of the "only if" statement. To prove the "if" statement, fix a preference profile $R$ satisfying assumptions (i) to (iv) and a finite partition $\mathcal{W} = \{W^1, ..., W^G\}$ with respect to which $R$ satisfies Agreement, Separability, and Internal Separability.

For any $g = 1, ..., G$, we consider, as in the proof of Proposition 1, the restriction $R_1^g$ of $R_1$to $\Omega^g$. If $W^g I_1 \emptyset$, it follows from assumption (i) that $R_1^g$ is representable by the constant map $u_1^g(A) = 0$ for all $A \in \Omega^g$, which is trivially a countably additive measure. Assume next that $W^g P_1 \emptyset$. Because of this assumption and (ii) to (iv), $R_1^g$ meets all the conditions of Theorem 3, Section 4, in Villegas (1964). It is therefore representable by a countably additive measure $u_1^g : \Omega^g \to \mathbb{R}_+$.

The rest of the proof is exactly the same as that of Proposition 1. ■

Whether Internal Separability should be part of an axiomatic definition of a commodity is, at best, disputable. We turn now to a condition that comes closer to our intuitive understanding of the concept. If $A \in \Omega$, a *division* of $A$ is a partition $\mathcal{A}$ of $A$ into subsets $A_1, ..., A_{|I|} \in \Omega$. It is *efficient* if no division $\mathcal{A}' = \{A_i'\}_{i \in I}$ of $A$ *Pareto dominates* it in the sense that $A_i' R_i A_i$ for all $i \in I$ and $A_i' P_i A_i$ for some $i \in I$.

*No Profitable Exchange.* For all $g = 1, ..., G$ and all $A \in \Omega^g$, every division of $A$ is efficient.

**Proposition 2.** *Assume $|I| \geq 2$. A finite partition $\mathcal{W}$ of the world $W$ is a system of cardinal commodities for the preference profile $R$ if and only if $R$ satisfies No Profitable Exchange and Separability with respect to $\mathcal{W}$.*

**Proof.** Again, we omit the straightforward verification of the "only if" statement. To prove the "if" statement, suppose $|I| \geq 2$, fix a preference profile $R$ satisfying assumptions (i) to (iv) and a finite partition $\mathcal{W} = \{W^1, ..., W^G\}$ with respect to which $R$ satisfies No Profitable Exchange and Separability. In view of the Lemma, it is enough to show that $R$ satisfies Internal Separability and Agreement. This is what we shall do. For the rest of the proof, fix $g \in \{1, ..., G\}$.

1) It is clear that for all $i \in I$,

$$\text{if } A, B, C \in \Omega^g \text{are pairwise disjoint, then } AR_iB \Leftrightarrow (A \cup C)R_i(B \cup C). \tag{3}$$

To check this claim, fix $i \in I$, say, $i = 1$, and pick $A, B, C \in \Omega^g$ such that $A \cap B = A \cap C = B \cap C = \emptyset$. Proceed by contradiction: suppose, first, that $AR_1B$ and $(B \cup C)P_1(A \cup C)$. If $BP_2A$, the division $\{A, B, \emptyset, ..., \emptyset\}$ of $A \cup B$ Pareto dominates $\{B, A, \emptyset, ..., \emptyset\}$ whereas if $AR_2B$, the division $\{B \cup C, A, \emptyset, ..., \emptyset\}$ of $A \cup B \cup C$ Pareto-dominates $\{A \cup C, B, \emptyset, ..., \emptyset\}$. In both cases, we obtain a contradiction to the No Profitable Exchange condition. Suppose next that $BP_1A$ and

$(A \cup C)R_1(B \cup C)$. If $AR_2B$, $\{B, A, \emptyset, ..., \emptyset\}$ Pareto dominates $\{A, B, \emptyset, ..., \emptyset\}$ and if $BP_2A$, $\{A \cup C, B, \emptyset, ..., \emptyset\}$ Pareto-dominates $\{B \cup C, A, \emptyset, ..., \emptyset\}$. Again, No Profitable Exchange is violated in both cases.

2) Next, we establish Internal Separability, namely, that for all $i \in I$,

$$\text{if } A, B, C \in \Omega^g \text{ and } C \cap (A \cup B) = \emptyset, \text{ then } AR_iB \Leftrightarrow (A \cup C)R_i(B \cup C).$$

Fix $i \in I$, say, $i = 1$, and pick $A, B, C \in \Omega^g$ such that $C \cap (A \cup B) = \emptyset$. Proceeding by contradiction, we distinguish again two cases.

In the first case, $AR_1B$ and $(B \cup C)P_1(A \cup C)$. Rewrite the first part of that statement under the form $((A\backslash B)\cup(A\cap B))R_1((B\backslash A)\cup(A\cap B))$. Since $A\backslash B$, $B\backslash A$, and $A \cap B$ are pairwise disjoint, it follows from (3) that $(A\backslash B)R_1(B\backslash A)$. If $(B \cup C)P_2(A \cup C)$, the division $\{A\backslash B, B \cup C, \emptyset, ..., \emptyset\}$ of $A \cup B \cup C$ Pareto dominates $\{B\backslash A, A \cup C, \emptyset, ..., \emptyset\}$, contradicting No Profitable Exchange. But if $(A \cup C)R_2(B \cup C)$, we may rewrite that statement as $((A\backslash B)\cup((A\cap B)\cup C))R_2((B\backslash A)\cup((A\cap B)\cup C))$. Since $A\backslash B$, $B\backslash A$, and $((A\cap B)\cup C)$ are pairwise disjoint, it follows from (3) that $(A\backslash B)R_2(B\backslash A)$. Therefore $\{B\cup C, A\backslash B, \emptyset, ..., \emptyset\}$ Pareto dominates $\{A\cup C, B\backslash A, \emptyset, ..., \emptyset\}$, contradicting No Profitable Exchange again.

In the second case, $BP_1A$ and $(A \cup C)R_1(B \cup C)$. An argument parallel to the one above yields similar contradictions to No Profitable Exchange.

3) Finally, we prove Agreement. Suppose, by way of contradiction, that there exist two agents, say 1 and 2, and sets $A, B \in \Omega^g$ such that $AR_1B$ and $BP_2A$. Since $A = (A\backslash B) \cup (A \cap B)$ and $B = (B\backslash A) \cup (A \cap B)$, Internal Separability, that we have just established, yields that $(A\backslash B)R_1(B\backslash A)$ and $(B\backslash A)P_2(A\backslash B)$. It follows that $\{A\backslash B, B\backslash A, \emptyset, ..., \emptyset\}$ Pareto dominates $\{B\backslash A, A\backslash B, \emptyset, ..., \emptyset\}$, contradicting No Profitable Exchange. ■

It is time to discuss the role of our assumptions. We should first stress that only (i) and (ii) are used in the proof of Proposition 1. That first proposition is true even if (iii) or (iv) are violated. In particular, it holds if $\Omega$ is finite.

Assumption (i) is crucial to all of our results. If it fails, any preferences over bundles that would be deduced from the original preferences over subsets of the world must fail to be monotone in the usual sense (spelled out in footnote 3). As a consequence, Agreement is no longer a *necessary* condition for reducing the preferences to orderings over bundles.

Assumption (ii) implies that the original preferences are orderings, but is stronger than the latter requirement. Of course, assuming that preferences are orderings is enough to obtain Proposition 1 in the case where $\Omega$ is finite. Assumption (ii) could presumably be derived from more basic hypotheses but we shall not attempt such a derivation here. Note, however, that no form of continuity is imposed on the assumed numerical representations of the preferences.

Assumptions (iii) and (iv) are used to obtain the cardinal reduction results only. In the proof of the Lemma, Internal Separability would not imply the additivity of $u_1^g$ if assumption (iii) were violated: the interested reader may consult Kraft, Pratt and Seidenberg (1959), section 4, for a finite counter-example. If (iv) is violated, countable additivity is not guaranteed.

We close this section by noting that mixed systems, where some commodities are ordinal and others are cardinal, are straightforward to define. The characterization of such systems poses no difficulty either: a partition $\mathcal{W} = \{W^1, ..., W^K, W^{K+1}, ..., W^G\}$ of the world is a mixed system of ordinal commodities $W^1, ..., W^K$ and cardinal commodities $W^{K+1}, ..., W^G$ if and only if the preference profile satisfies Separability with respect to $\mathcal{W}$, Agreement with respect to $\{W^1, ..., W^K\}$, and either i) Agreement and Internal Separability or ii) No Profitable Exchange (if $|I| \geq 2$) with respect to $\{W^{K+1}, ..., W^G\}$.

## 3. An Example

The example discussed in this section serves two purposes. Our first goal is merely to illustrate, clarify, and contrast the two definitions of a commodity proposed in Section 2. Our second goal is somewhat biased. Since the concept of a cardinal commodity is well known to economists and has proved so useful, it really needs no defense. Because the notion of an ordinal commodity is certainly less familiar, we would like to show that it may be a helpful and natural modeling tool. In the example below, we will be able to reduce the original preferences of the agents to preferences over ordinal bundles or to preferences over cardinal bundles of *differently defined* commodities. The example is designed to show that an ordinal reduction is sometimes more appealing than a cardinal one.

The set of agents, $I$, consists of two large firms, 1 and 2. There are a thousand *identical* skilled workers indexed by $w \in L^1 = \{1, ..., 1000\}$ and a thousand identical unskilled workers indexed by $w \in L^2 = \{1001, ..., 2000\}$. Workers are not agents. Each worker $w$ is endowed with an interval of one unit of perfectly divisible time, $E_w$. The world is the union of these intervals, $W = \cup_{w=1}^{2000} E_w$. Firm $i$'s preference relation over (the $\sigma$-algebra of Borel) subsets of $W$ admits the numerical representation $u_i : \Omega \to \mathbb{R}$,

$$u_i(A) = v_i\Big(\sum_{w \in L^1} \sqrt{\lambda(A \cap E_w)}, \sum_{w \in L^2} \sqrt{\lambda(A \cap E_w)}\Big),$$

where $\lambda$ is Lebesgue measure and $v_i$ is an arbitrary monotone function defined on $\mathbb{R}_+^2$. We further assume that the firms' preferences differ: $u_1$ is not a monotone transformation of $u_2$. The key feature of this example is that firms do not care merely about the number of hours of skilled and unskilled labor they hire; they also have a preference for part-time over full-time workers. For instance, any firm prefers to hire the skilled workers 1 and 2 for half an hour each rather than worker 1 for an hour.

How many commodities are there in this world? One answer is: two ordinal commodities, namely, $W^1 = \cup_{L^1} E_w$ and $W^2 = \cup_{L^2} E_w$, which may naturally be called "skilled and unskilled labor". Another answer is: two thousand cardinal commodities, namely, $E_1, ..., E_{2000}$. It follows directly from either the Lemma or Proposition 2 in Section 2 that no partition of the world into fewer than 2000 subsets constitutes a system of cardinal commodities. Since all workers in a given group, say, $E_1$, are identical, it seems odd and counter-intuitive to consider the labor time that each of them supplies as a distinct and separate commodity. It is only our insistence on providing a cardinal measure of all commodities that forces us to this artifact.

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