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Overcoming Natural Resource Constraints Through R&D

Jean-Pierre Amigues^{}, Ngo Van Long[†], Michel Moreaux[‡]*

Résumé / Abstract

Nous étudions la politique optimale en R&D dans le secteur de ressources naturelles. On distingue deux cas : ressources non renouvelables, et ressources renouvelables. Dans le premier cas, nous montrons qu'il est utile de construire un indice de rareté, qui est le produit du niveau de connaissance scientifique et du stock de ressources. Pourvu que le taux d'escompte ne soit pas trop élevé, il existe un niveau critique de cet indice au-dessous duquel il faut maximiser le taux d'investissement en R&D. À partir de ce niveau critique, on peut atteindre un état stationnaire de consommation en substituant la ressource par la connaissance. Dans le cas de ressources renouvelables, la politique optimale est d'accorder la priorité à la production des biens de consommation, et les investissements en R&D sont déterminés comme résiduels.

Mots clés : politique optimale en R&D, ressources naturelles, indice de rareté.

We study the optimal policies of research and development in the context of a resource-exploiting economy. We distinguish two cases: non-renewable resources and renewable resources. In the first case, we show that it is useful to construct an index of scarcity, which is the product of the level of technical know-how and the aggregate stock of resources. Provided that the rate of discount is not too high, there exists a critical level of this index, below which one must maximize the rate of investment in R&D. Starting from this critical level, it is possible to maintain a constant rate of consumption, by substituting knowledge for natural resources. In the case of renewable resources, we show that the optimal policy is to give priority to the production of consumption goods, and the rates of investments in R&D are determined residually.

Keywords: *optimal R&D policies, natural resources, scarcity index.*

Codes JEL : Q20, Q30, D90

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1 Introduction

The industrial revolution has been characterized by some historians as an energetic switch from the use of the diffuse energy of the sun via either biological transformations (wood, animals and humans) or thermo-mechanical transformations (wind, hydro power), the so called *organic societies* according to the Wrigley¹ terminology, to the use of fossile energy resources (coal initially), the *mineral based energy economy* (ibid).

Besides the advance of fundamental science there has been, and there is, a lot of more applied and dedicated research aiming at improving the productivity of the most scarce factors and particularly the energy factors. But there is a big difference between improving the efficiency of the use of a renewable resource and improving the efficiency of a non renewable resource. In the first case this improvement generates a permanent flow of gains, limited at each point of time by the available flow of the resource, and, as far as energy is concerned², by poor direct intertemporal transfers of these gains. In the second case, the gain is proportional to the stock of the non renewable resource, which is finite, but can be used at any point of time. We examine in this study some implications of this difference for the optimal dedicated R&D policies aiming at overcoming these natural resource constraints.

There exist specific factor scarcities if and only if the inputs of the production process are not perfect substitutes³. We assume here the extreme case of a Leontief technology in which labor and resource are strictly complementary in the production of some aggregate consumption good and we

¹See Wrigley (1988). There exists an avalanche of books and studies about the history of technology and science, and some encyclopedic monuments like the *Oxford History of Technology* by Singer and al. (abridged in Derry and Williams, 1960), the French *Histoire générale des techniques* under the editorship of Daumas, or the Forbes *Studies in Ancient Technology*. More modest (by the size) though rich panoramas with strongly different perspectives can be found in Basalla (1988), Kirby and al. (1956), Mokyr (19990.a, 1990.b and 1993), Usher (1954) and for the present times in Ruttan (2001). Admittedly, such a selection is partly, some maybe would say totally, arbitrary.

²Although we have mainly in mind energy resources, there are many goods which can be produced either from renewable resources or from non renewable resources. For example, the houses can be build either from wood or from iron and stones, furniture either from wood or iron and steel, plastic either from corn or coal and petrol, etc...

³As pointed out some time ago by Dasgupta and Heal (1979, p207, last paragraph) : "The Cobb-Douglas economy merits special attention. The point to note is that it is not possible to distinguish between capital, resource and labour augmenting technical progress".

assume that the technical coefficient of the resource input can be increased through a R&D investment. Such an extreme assumption about the production function allows for the most contrasting view of the effects of a productivity improving R&D policy upon a renewable resource based economy or a non renewable resource based economy. Voluntary disregarding the material capital accumulation problems allows to exacerbate the difference between the two resources concerning the easiness of intertemporal transfers of the productivity gains generated by the R&D effort. However note that such transfers are not totally forbidden even in the renewable resource economy since there exists a dedicated knowledge capital through which such transfers can be made.

Rather than studying how to switch from one type of economy to the other one⁴, we determine the optimal policy in the both cases in which the resource is a renewable resource and in which the resource is a non renewable resource. As we shall see the optimal policies are strongly contrasted.

The paper is organized as follows. The model is developed in section 2. The non renewable resource economy is examined in section 3 and the renewable resource economy in section 4. We briefly conclude in section 5.

2 The model

We consider two types of economies. In the first type some aggregate consumption good can be produced from labor and some non renewable or exhaustible resource whereas in the second one the natural resource is a renewable resource. In both economies the population is stationary and the labor supply is inelastic. Without loss of generality we may assume that the constant labor supply is equal to 1.

In this study the both factors, labor and natural resource, are assumed to be strictly complementary in the consumption good production sector. Let l_{et} and l_{rt} be the instantaneous levels of the labor inputs in the economy using the exhaustible and the renewable resources respectively, s_t and r_t the instantaneous levels of the non renewable and renewable resource inputs and c_t the corresponding instantaneous production level of the consumption good. Then :

⁴See for example Bretschger and Smulders (2003), De Nooij and Smulders (2003), Thavonon and Salo (2001) and Tsur and Zemel (2000, 2003).

– in the exhaustible resource economy :

$$c_t = \min\{A_e \ell_{et}, B_{et} s_t\} \quad , t \geq 0 \quad (2.1.e)$$

– in the renewable resource economy :

$$c_t = \min\{A_r \ell_{rt}, B_{rt} r_t\} \quad , t \geq 0 \quad (2.1.r)$$

In each type of economy the technical coefficients of labor, A_e and A_r , are assumed to be fixed over time. On the contrary the technical coefficients of the resources can be improved through a dedicated R&D effort. Let us denote by n_{et} and n_{rt} this part of the labor supply devoted to improving B_{et} and B_{rt} respectively. B_{et} and B_{rt} may be seen as stocks of scientific and technical knowledge, cumulatively increasing, the growth rates of which are higher the higher are the labor inputs devoted to R&D, that is respectively⁵ :

$$\dot{B}_{et} = b_e n_{et} B_{et}, \text{ and } \dot{B}_{rt} = b_r n_{rt} B_{rt}.$$

This assumption about the possibility of an indefinitely growing efficiency in the use of natural resources could seem very strong. But note that, although the quantity of resource needed by unit of consumption good tends asymptotically towards zero provided that $n_{et} \in [\underline{n}_e, 1]$, $\underline{n}_e \in (0, 1]$ or $n_{rt} \in [\underline{m}_r, 1]$, $\underline{m}_r \in (0, 1]$, it is always strictly positive at any point of time so that it is never possible to produce without resource. Another reason for assuming such an efficient research sector is that it permits to simplify drastically the analysis, like the assumption of a labor force growing at a constant positive rate in the growth theory. The specification adopted here is the most simple amongst those generating a stationary state in the non renewable resource economy. The crucial point to get a sustainable consumption path⁶ in the non renewable resource economy is that we could have $\dot{B}_{et}/B_{et} \geq \underline{g}_B$, where \underline{g}_B is some strictly positive constant, for some research effort path $n_{et} = n \in (0, 1)$.

For the sake of simplicity we assume that the labor can be instantaneously and freely transferred from the consumption good production sector to the research sector and vice versa. Hence the only constraints on the uses of labor are :

$$1 - \ell_{et} - n_{et} \geq 0 \quad , \quad \ell_{et} \geq 0, \text{ and } n_{et} \geq 0 \quad , \quad t \geq 0$$

⁵It can be shown that this form of technological knowledge production function is the only one generating a steady sustainable path in the non renewable economy.

⁶that is a consumption path $C : \{c_t \geq \underline{c} > 0, t \geq 0\}$.

$$1 - \ell_{rt} - n_{rt} \geq 0, \ell_{rt} \geq 0, \text{ and } n_{rt} \geq 0, t \geq 0$$

In the economy using the non renewable resource there exists initially some stock S_0 of the exhaustible resource. We assume that the extraction cost is equal to 0. Equivalently (2.1.e) may be seen as the production function of an integrated process the primary inputs of which are labor, resource and technical knowledge capital. Thus the dynamics of the available resource stock S_t is given by :

$$\dot{S}_t = -s_t, S_t \geq 0, \text{ and } s_t \geq 0, t \geq 0.$$

In the economy using the renewable resource there exists some instantaneous natural flow of resource denoted by \bar{r} . The resource cannot be stored or equivalently its storage cost is prohibitively high in the long run⁷, so that this part of the flow which is not immediately used in the consumption good production is definitively lost. As for the exhaustible resource we do not distinguish pure extraction costs from the other costs, thus(2.1.r) may be understood as the production function of an integrated process.

The constraints on r_t are :

$$\bar{r} - r_t \geq 0, \text{ and } r_t \geq 0.$$

The instantaneous utility of consumption $u(c_t)$ is proportional to the instantaneous rate of consumption c_t . Without loss of generality we may assume that the proportionality coefficient is equal to 1. The social welfare $W(C)$ generated by any consumption path $C = \{c_t, t \geq 0\}$ is equal to the sum of the discounted instantaneous utilities at some constant discount rate $\rho > 0$, that is :

$$W(C) = \int_0^{\infty} c_t e^{-\rho t} dt.$$

Since either $c_t \leq A_e$ or $c_t \leq A_r$ then this integral is well defined for any feasible path C . In what follows, for any consumption path C , we shall denote by $W_t(C)$ the present value at time 0, of that tail of the consumption path beginning at time t :

$$W_t(C) = \int_t^{\infty} c_{\tau} e^{-\rho \tau} d\tau, t \geq 0.$$

⁷Daily details are omitted.

In each type of economy the benevolent social planner maximizes $W(C)$ under the constraints corresponding to the type of resource used in the economy. Noting that since any optimal program has to be efficient, we may restrict the set of admissible paths to the set of efficient programs, that is :

- in the exhaustible resource economy, to those programs such that :

$$A_e \ell_{et} = B_{et} s_t \Rightarrow s_t = A_e B_{et}^{-1} \ell_{et}$$

- in the renewable resource economy, to those programs such that :

$$A_r \ell_{rt} = B_{rt} r_t \Rightarrow r_t = A_r B_{rt}^{-1} \ell_{rt}$$

Hence in the exhaustible resource economy the social planner program may be formulated as the following problem (*PE*) :

$$(PE) \quad \max_{\{(\ell_{et}, n_{et}), t \geq 0\}} \int_0^{\infty} A_e \ell_{et} e^{-\rho t} dt \quad (PE.0)$$

$$\dot{S}_t = -A_e B_{et}^{-1} \ell_{et}, \quad S_t \geq 0, t \geq 0, \text{ and } S_0 \text{ given} \quad (PE.1)$$

$$\dot{B}_{et} = b_e n_{et} B_{et}, \quad t \geq 0, B_{e0} \text{ given} \quad (PE.2)$$

$$1 - \ell_{et} - n_{et} \geq 0, \ell_{et} \geq 0, \text{ and } n_{et} \geq 0, t \geq 0 \quad (PE.3)$$

In the renewable resource economy the social planner program is the following problem (*PR*) :

$$(PR) \quad \max_{\{(\ell_{rt}, n_{rt}), t \geq 0\}} \int_0^{\infty} A_r \ell_{rt} e^{-\rho t} dt \quad (PR.0)$$

$$\bar{r} - A_r B_{rt}^{-1} \ell_{rt} \geq 0, t \geq 0 \quad (PR.1)$$

$$\dot{B}_{rt} = b_r n_{rt} B_{rt}, t \geq 0, \text{ and } B_{r0} \text{ given} \quad (PR.2)$$

$$1 - \ell_{rt} - n_{rt} \geq 0, \ell_{rt} \geq 0, \text{ and } n_{rt} \geq 0, t \geq 0 \quad (PR.3)$$

3 The non renewable resource economy

In the non renewable resource economy two cases have to be distinguished. If first the productivity of the research effort b_e is higher than the social rate

of discount ρ , then there exists some critical locus L in the state variable plane (B_e, S) , $L = \{(B_e, S) : B_e S = \bar{K}\}$ where \bar{K} is determined by the structural coefficients of the model, from which the unique optimal stationary consumption and sectoral employment path can start. Then, if $B_{e_0} S_0 \neq \bar{K}$, either $B_{e_0} S_0 > \bar{K}$, or $B_{e_0} S_0 < \bar{K}$. For $B_{e_0} S_0 > \bar{K}$ the optimal policy is to consume at the maximum feasible rate and do not invest in research as long as $B_{e_0} S_t > \bar{K}$, and once $B_{e_0} S_t = \bar{K}$ then switch to the optimal stationary path along which the research effort and the consumption rate are constant both at a strictly positive level. For $B_{e_0} S_0 < \bar{K}$ the optimal policy is to catch up the L locus as soon as possible that is to allocate all the available labor to the development of the scientific and technological knowledge basis in order to enjoy the optimal stationary consumption rate as soon as possible. But if the efficiency of the knowledge production process, b_e , is lower than the social rate of discount, ρ , then the welfare balance sheet of any effort aiming at boosting the resource productivity coefficient B_e , would be in the red. In those societies in which increasing the technical knowledge corpus would require a too hard effort, the optimal policy shrinks down to solving a classical pure cake eating problem.

3.1 General characteristics of the optimal policies

Let LE_t be the Lagrangian of the program (PE) ⁸ :

$$LE_t = A_e \ell_{et} e^{-\rho t} - \lambda_t A_e B_{et}^{-1} \ell_{et} + \mu_{et} b_e n_{et} B_{et} + \omega_{et} [1 - \ell_{et} - n_{et}] + \gamma_{el,t} \ell_{et} + \gamma_{en,t} n_{et}.$$

The first order conditions are :

$$\partial LE_t / \partial \ell_{et} = 0 \Leftrightarrow A_e e^{-\rho t} - \lambda_t A_e B_{et}^{-1} - \omega_{et} + \gamma_{el,t} = 0 \quad (3.1)$$

$$\partial LE_t / \partial n_{et} = 0 \Leftrightarrow \mu_{et} b_e B_{et} - \omega_{et} + \gamma_{en,t} = 0 \quad (3.2)$$

together with the complementary slackness conditions :

$$\omega_{et} \geq 0, 1 - \ell_{et} - n_{et} \geq 0, \text{ and } \omega_{et} [1 - \ell_{et} - n_{et}] = 0 \quad (3.3)$$

$$\gamma_{el,t} \geq 0, \ell_{et} \geq 0, \text{ and } \gamma_{el,t} \ell_{et} = 0 \quad (3.4)$$

⁸Since we shall restrict the investigation to the set of efficient programs for which $S_t \geq 0, t \geq 0$, this constraint is deleted in the expression of the lagrangian, as usual in this type of program.

$$\gamma_{en,t} \geq 0, n_{et} \geq 0, \text{ and } \gamma_{en,t}n_{et} = 0 \quad (3.5)$$

The dynamics of the costate variables λ_t and μ_{et} are given by :

$$\dot{\lambda}_t = -\partial LE_t/\partial S_t \Leftrightarrow \dot{\lambda}_t = 0 \Leftrightarrow \lambda_t = \lambda = cte \quad (3.6)$$

$$\dot{\mu}_{et} = -\partial LE_t/\partial B_{et} \Leftrightarrow \dot{\mu}_{et} = -\lambda A_e B_{et}^{-2} \ell_{et} - \mu_{et} b_e n_{et}. \quad (3.7)$$

Last the transversality conditions at infinity are :

$$\lim_{t \uparrow +\infty} \lambda S_t = 0, \text{ and } \lim_{t \uparrow +\infty} \mu_{et} B_{et} = 0. \quad (3.8)$$

The above conditions imply that, at any point at time t , there exist two ways to decompose the discounted value $W_t(C)$ of an optimal consumption plan C , and that the imputed value of the resource stock λS_t must be equal to the imputed value of the stock of knowledge $\mu_{et} B_{et}$.

The first way proceeds from the optimality condition (3.1), the complementary slackness condition (3.4) and the transversality condition (3.8) relative to the asymptotic value of S_t . Since, by (3.4), $\gamma_{el,t} \ell_{et} = 0$, then multiplying (3.1) by ℓ_{et} we get :

$$A_e \ell_{et} e^{-\rho t} - \lambda A_e B_{et}^{-1} \ell_{et} - w_{et} \ell_{et} = 0,$$

that is :

$$c_t e^{-\rho t} = \lambda s_t + w_{et} \ell_{et}.$$

By the transversality condition either $\lambda = 0$, or $\lambda > 0$ and then $\lim_{t \uparrow +\infty} S_t = 0$ so that $S_t = \int_t^\infty s_\tau d\tau$, hence the first decomposition :

$$W_t(C) = \int_t^\infty c_\tau e^{-\rho \tau} d\tau = \lambda S_t + \int_t^\infty w_{e\tau} \ell_{e\tau} d\tau, \quad t \geq 0.$$

For the second way, let us start from (3.7), which, multiplied by B_{et} , gives :

$$\dot{\mu}_{et} B_{et} = -\lambda A_e B_{et}^{-1} \ell_{et} - \mu_{et} b_e n_{et} B_{et},$$

that is :⁹

$$(\mu_{et} \dot{B}_{et}) = -\lambda s_t,$$

⁹For any time functions x_t and y_t , we denote by $(x_t \dot{y}_t)$ the time derivative of their product, that is $(x_t \dot{y}_t) = \dot{x}_t y_t + x_t \dot{y}_t$.

hence :

$$\int_t^\infty (\mu_{e\tau} \dot{B}_{e\tau}) d\tau = \mu_{e\infty} B_{e\infty} - \mu_{et} B_{et} = -\lambda \int_t^\infty s_\tau d\tau = -\lambda S_t.$$

By the transversality condition $\mu_{e\infty} B_{e\infty} = 0$, so that :

$$\mu_{et} B_{et} = \lambda S_t \quad , \quad t \geq 0, \quad (3.9)$$

and :

$$W_t(C) = \mu_{et} B_{et} + \int_t^\infty w_{e\tau} \ell_{e\tau} d\tau \quad , \quad t \geq 0.$$

Proposition 1 In the non renewable resource economy, along the optimal path, at each point of time, the imputed value of the resource stock must be equal to the imputed value of the technological knowledge stock, and the discounted value of the optimal consumption plan C is given by :

$$W_t(C) = \lambda S_t + \int_t^\infty w_{e\tau} \ell_{e\tau} d\tau = \mu_{et} B_{et} + \int_t^\infty w_{e\tau} \ell_{e\tau} d\tau, t \geq 0.$$

3.2 Optimal stationary paths

Let us consider the optimal paths, if any, along which both $\ell_{et} > 0$ and $n_{et} > 0$ so that $\gamma_{el,t} = \gamma_{en,t} = 0$, and $1 - \ell_{et} - n_{et} = 0$ and let us show that such paths are necessarily the unique optimal regular or stationary path along which the consumption rate can be indefinitely maintained constant at some positive level.

Multiplying the both sides of (3.7) by B_{et} , we obtain :

$$(\mu_{et} \dot{B}_{et}) = -\lambda A_e B_{et}^{-1} \ell_{et}. \quad (3.10)$$

On the other hand, from (3.1) and (3.2) with $\gamma_{el,t} = \gamma_{en,t} = 0$, we get :

$$b_e \mu_{et} B_{et} = A_e e^{-\rho t} - \lambda A_e B_{et}^{-1},$$

hence :

$$b_e (\mu_{et} \dot{B}_{et}) = -\rho A_e e^{-\rho t} + \lambda A_e B_{et}^{-2} \dot{B}_t.$$

Substituting for \dot{B}_{et} its value given by (PE.2) and noting that $n_{et} = 1 - \ell_{et}$, this last equation may be written as :

$$(\mu_{et} \dot{B}_{et}) = -\rho b_e^{-1} A_e e^{-\rho t} + \lambda A_e B_{et}^{-1} (1 - \ell_{et}).$$

Last substituting for $(\mu_{et} B_{et})$ its expression (3.10), we get the value of B_{et} over any time interval during which $\ell_{et} > 0, n_{et} > 0$ and $1 - \ell_{et} - n_{et} = 0$:

$$B_{et} = \lambda b_e \rho^{-1} e^{\rho t}. \quad (3.11)$$

Time differentiating (3.11) and taking (PE.2) into account results in :

$$b_e n_{et} B_{et} = \lambda b_e e^{\rho t}$$

and since $\lambda b_e e^{\rho t} = \rho B_{et}$ according to (3.11), then :

$$n_{et} = \rho b_e^{-1}. \quad (3.12)$$

Since we must have $n_{et} < 1$, then there exists an optimal path with both $\ell_{et} > 0$ and $n_{et} > 0$ and full employment of the labor supply, iff the productivity of the research effort b_e is higher than the social rate of discount ρ .

Let us assume that from T onwards $n_{et} = \rho b_e^{-1}$ so that :

$$B_{et} = B_{eT} e^{\rho(t-T)}, \quad t \geq T. \quad (3.13)$$

Since $\ell_{et} = 1 - n_{et}$, then the instantaneous extraction rate of the resource must be equal to :

$$s_t = A_e B_{eT}^{-1} (b_e - \rho) b_e^{-1} e^{-\rho(t-T)}, \quad t \geq 0. \quad (3.14)$$

For any $t \geq T$, let \bar{S}_{Tt} be the quantity of resource extracted over the time interval $[T, t)$:

$$\bar{S}_{Tt} = \int_T^t s_\tau d\tau = A_e B_{eT}^{-1} (b_e - \rho) (\rho b_e)^{-1} [1 - e^{-\rho(t-T)}], \quad (3.15)$$

so that :

$$\bar{S}_{T\infty} = \lim_{t \uparrow +\infty} \bar{S}_{Tt} = A_e B_{eT}^{-1} (b_e - \rho) (\rho b_e)^{-1}. \quad (3.16)$$

We conclude that, in order to follow indefinitely the optimal stationary path from T onwards, the product $K_t \equiv B_{et} S_t$ must be equal to the critical value \bar{K} at time T :

$$K_T = B_{eT} S_T = A_e (b_e - \rho) (\rho b_e)^{-1} \equiv \bar{K}. \quad (3.17)$$

It will be noticed that K_t remains constant at the level \bar{K} along this path¹⁰.

More generally $K_t = B_{et}S_t$ may be seen as the amount of consumption good which could be produced from the resource stock S_t , would the productivity of the resource remain fixed at its level B_{et} at time t . Thus K_t could be understood as the consumption potential at time t . Along any stationary path this consumption potential has to be held constant, and along the optimal stationary path the consumption potential has to be held constant at the level \bar{K} .

Let us complete the characterization of the optimal stationary path. The instantaneous consumption rate is constant and given by (cf (3.13) and (3.14)) :

$$c_t = B_{et}s_t = A_e(b_e - \rho)b_e^{-1}. \quad (3.18)$$

Last from (3.2) (or (3.1)), (3.9), (3.11) and (3.13)) we get, for $t > T$:

$$\lambda = \rho b_e^{-1} B_{eT} e^{-\rho T}, \mu_{et} = \rho b_e^{-1} S_T e^{-\rho(2t-T)} \text{ and } w_{et} = \rho \bar{K} e^{-\rho t} \quad (3.19)$$

where B_{eT} and S_T are such that $B_{eT}S_T = \bar{K}$.

We conclude as follows.

Proposition 2 *In the non renewable resource economy, along an optimal trajectory, there may exist a time interval during which the employment is positive in the both sectors of the economy iff the productivity in the research sector b_e is higher than the social rate of discount ρ . In this case $\ell_{et} = (b_e - \rho)b_e^{-1}$, $n_{et} = \rho b_e^{-1}$ and $c_t = A_e(b_e - \rho)b_e^{-1}$. This optimal stationary path can be indefinitely sustained from T onwards provided that $K_T \equiv B_{eT}S_T = \bar{K} \equiv A_e(b_e - \rho)(\rho b_e)^{-1}$.*

Note that the optimal constant consumption stream is an increasing function of the labor productivity index b_e in the research sector as it could have been expected but a decreasing function of the social rate of discounts although the employment in the research sector is increasing with ρ . We have

¹⁰For $t \geq T$, we get : $B_{et}S_t = B_{et}[S_T - \bar{S}_{Tt}]$. Substituting for B_{et} its value given by (3.13) and for \bar{S}_{Tt} its value given by (3.15), we obtain :

$$\begin{aligned} B_{et}S_t &= B_{eT}S_T e^{\rho(t-T)} - A_e(b_e - \rho)(b_e \rho)^{-1} e^{\rho(t-T)} + A_e(b_e - \rho)(b_e \rho)^{-1} \\ \text{given that } A_e(b_e - \rho)(\rho b_e)^{-1} &= B_{eT}S_T \text{ by (3.17), we obtain :} \\ B_{et}S_t &= B_{eT}S_T e^{\rho(t-T)} - B_{eT}S_T e^{\rho(t-T)} + B_{eT}S_T = B_{eT}S_T. \end{aligned}$$

got the same result in a model in which the utility function is a constant intertemporal elasticity of substitution (CIES) function and both production and research sides of the model are like the present ones (see Amigues-Grimaud-Moreaux (2003-a)). We have got also the same result in a model in which the utility function is a CIES function, the consumption good production function is a CES function and the resource productivity research sector is like the present one (see Amigues-Grimaud-Moreaux (2003-b)). Furthermore if A_e can be improved too, by a dedicated research effort, then the consumption growth rate along the optimal balanced path is a decreasing function of the social rate of discount (see Amigues and Moreaux (2003)).

It must be pointed out that the research effort n_t , hence the employment in the consumption sector, does not depend upon the shape of the utility function in an optimal stationary state. This is also the case for a CES production function, as shown in Amigues-Grimaud-Moreaux (2003-b), provided that the resource technical coefficient improving process be the same, that is $\dot{B}_{et} = bn_{et}B_{et}$. Then the optimal research effort is also the same, that is $n_{et} = \rho b_e^{-1}$. If both A_e and B_{et} can be increased through dedicated research efforts, then along a balanced growth path, the employment in the resource productivity research sector does not depend upon the utility parameters whereas the employment in the labor productivity research sector is an increasing function of the elasticity of intertemporal substitution (see Amigues and Moreaux (2003)). However, in all the above cases, the research effort in the resource productivity sector along the optimal path towards the stationary or balanced trajectory is sensitive to the type and parameters of the utility function that the social planner maximizes.

Another characteristic of the present model which is worth to emphasize, is that the so called Hartwick rule does not hold. In the present context of strict factor complementarity between labor and a "technical knowledge and resource" aggregate factor, we may not use the original framework laid down by Hartwick. So let us start from the following specification of the rule given by Hartwick himself in Hartwick (1989)¹¹ : "..., if current net value of resources used up at time t is used to purchase new reproducible capital goods at current prices of date t , consumption can be maintained constant. What the generation at date t uses up in terms of resources is passed one dollar for dollar as additional buildings and machines". There the only capital is the dedicated technological knowledge, the shadow price of which is μ_{et} . Thus the value of the capital gross investment is equal to $\mu_{et}\dot{B}_{et}$ and the value of the net investment is equal to $\mu_{et}\dot{B}_{et} + \dot{\mu}_{et}B_{et}$. The value of the resource

¹¹cf Hartwick (1989)p101

rent is given by $\lambda_t s_{et}$. Since $\lambda_t S_t = \mu_{et} \dot{B}_{et}$ according to (3.9) and $\lambda_t = \lambda$, a constant, then $-\lambda_t s_{et} = \mu_{et} \dot{B}_{et} + \dot{\mu}_{et} B_{et}$, that is the value of the resource rent is equal neither to the gross nor to net value of the capital investment.

3.3 Optimal policies

It remains to determine what are the optimal policies first if $b_e > \rho$ but $K_0 \neq \bar{K}$ and second if $b_e < \rho$.

3.3.1 Case $b_e > \rho$ and $K_0 \neq \bar{K}$

If $b_e > \rho$ and $K_0 > \bar{K}$ the consumption potential is initially high, that is the resource stock and/or the technical knowledge capital are abundant. The linearity of the instantaneous utility function together with a positive social rate of discount both suggest that initially the research effort has to be delayed. The optimal policy would be first to consume at the maximal rate, $\ell_{et} = 1$ and $n_{et} = 0$, as long as $K_t > \bar{K}$. Since $s_t = A_e B_{e0}^{-1} = cte$ during this initial period, there exists some time T at which $B_{e0} S_T = \bar{K}$. Then, from T onwards, the economy can follow the optimal stationary path. We show in Appendix (§6.1.1) that such a policy is indeed the optimal policy.

If $b_e > \rho$ but $K_0 < \bar{K}$ a symmetrical argument leads to the conclusion that the economy must first invest in technical knowledge at the maximal rate, that is $\ell_{et} = 0$ and $n_{et} = 1$, as long as $K_t < \bar{K}$. During this initial phase $B_{et} = B_{e0} e^{b_e t}$ and there exists some time T at which $B_{eT} S_0 = \bar{K}$. Next the economy can go along the optimal stationary path. We show in Appendix (§6.1.2) that this policy is the true optimal policy.

Clearly this is a case of most rapid approach to the optimal stationary state (see Spence and Starett (1975), and Tsur and Zemel (2001,2003)) and we conclude as follows.

Proposition 3 In the non renewable economy, if the productivity in the research sector \bar{b}_e is higher than the social rate of discount ρ , then :

i- either $K_0 = \bar{K}$ and the optimal policy is to go immediately along the optimal stationary path ;

ii- or $K_0 \neq \bar{K}$ and the optimal policy is to rejoin as soon as possible the optimal stationary path, that is if $K_0 > \bar{K}$, to consume at the maximal rate ($\ell_{et} = 1$), and if $K_0 < \bar{K}$ to expand the technical knowledge stock at the maximal rate ($n_{et} = 1$), and next go along the optimal stationary path.

The optimal path in the (c, K) - plane is illustrated in Figure 1 below¹² .

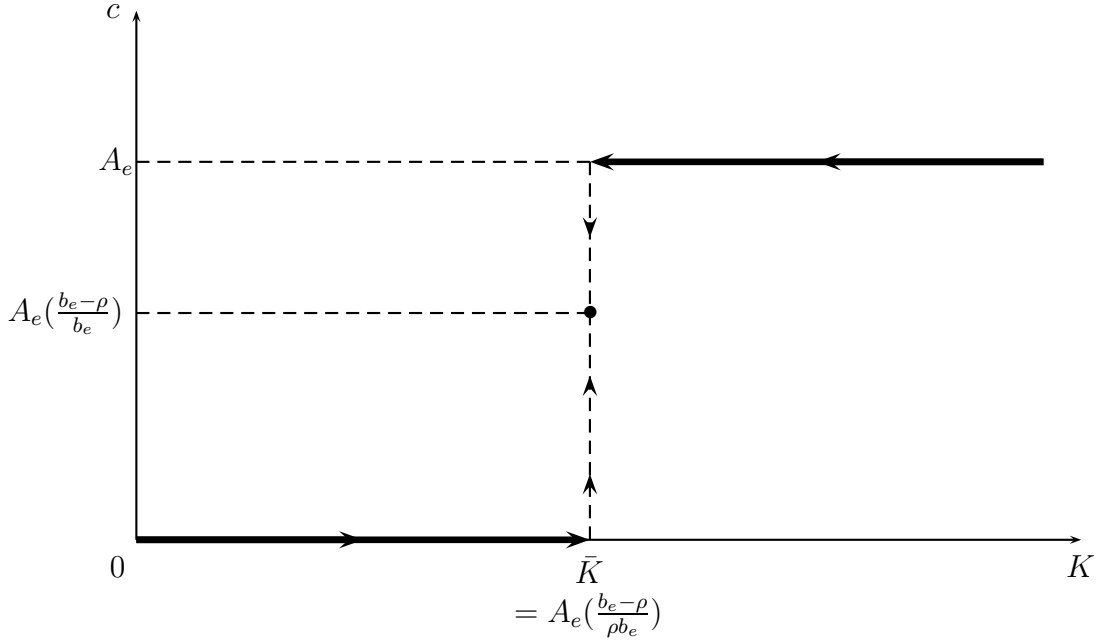


Figure 1 - Optimal path in the non renewable resource economy.
Case $b_e > \rho$

3.3.2 Case $b_e < \rho$

Since the productivity of the resource effort is low, we mean lower than the social rate of discount, one suspects it is never optimal to invest. Let us show that it is the right policy.

For any knowledge capital B_e and any stock of resource S , if the society is consuming at the maximal rate $c = A_e$ then $s = A_e B_e^{-1}$ so that the length Δ of the consumption period is equal to :

$$\Delta = A_e^{-1} B_e S$$

If t_1 is the time at which the consumption period is beginning, the discounted value of the consumption plan, W_{t_1} , is given by :

¹²Assuming that the instantaneous utility function is a CIES function would result in a smooth convergence towards the same regular path (see Amigues, Grimaud and Moreaux, 2003.a). Furthermore the result of Proposition 3 and the result of the following Proposition 4 are robust to less extreme assumptions about the production function and still hold with CES production functions (see Amigues, Grimaud and Moreaux, 2003.b).

$$W_{t_1} = \int_{t_1}^{t_1+\Delta} A_e e^{-\rho t} dt = \rho^{-1} A_e [1 - e^{-\rho\Delta}] e^{-\rho t_1}.$$

We know from Proposition 1 that $b_e < \rho$ implies that either $\ell_{et} = 1$ or $n_{et} = 1$ since $\ell_{et} = n_{et} = 0$ cannot be optimal if $S_t > 0$. Let us assume that over the initial period $[0, t_1)$ the society invests in knowledge capital, $n_{et} = 1$ and next the society consumes at the maximal rate as long as the resource is not exhausted. At time t_1 , $B_{et_1} = B_{e0} e^{b_e t_1}$ and the length of the consumption period is equal to $\Delta(t_1)$:

$$\Delta(t_1) = A_e^{-1} B_{e0} S_0 e^{b_e t_1},$$

so that :

$$W(t_1) = \rho^{-1} A_e [1 - e^{-\rho\Delta(t_1)}] e^{-\rho t_1}.$$

Differentiating W_{t_1} with respect to t_1 we get :

$$dW_{t_1}/dt_1 = e^{-\rho t_1} [-A_e + A_e e^{-\rho\Delta(t_1)} + b_e B_{e0} S_0 e^{-\rho\Delta t_1} e^{b_e t_1}].$$

The sign of the derivative is given by the sign of the term in brackets, denoted by M . For $t_1 = 0$:

$$M = -A_e + [A_e + b_e K_0] e^{-\rho(A_e^{-1} K_0)}.$$

Note that $K_0 = 0$ implies that $M = 0$. Let N be the second term of the right hand side of the above equality. Then :

$$dN/dK_0 = [(b_e - \rho) - \rho b_e A_e^{-1} K_0] e^{-\rho(A_e^{-1} K_0)}.$$

From $b_e < \rho$, we conclude that $dN/K_0 < 0$, so that :

$$K_0 > 0 \Rightarrow M < 0 \Rightarrow \frac{dW_{t_1}}{dt_1} \Big|_{t_1=0} < 0.$$

Last since $t_1 = 0$ and $K_0 > 0$ are both arbitrary, it is never optimal to invest. We show in Appendix (§6.1.3) that, for this policy, all the optimality conditions (3.1)-(3.8) are satisfied.

Proposition 4 In the non renewable resource economy, if the productivity in the research sector b_e is lower than the social rate of discount ρ , then the

optimal policy is to consume at the maximal rate from the start as long as the resource is not exhausted, and never invest.

The strong discounting in the optimal consumption path, from the constant consumption path A_e abruptly down to o once the resource stock is exhausted, would be forbidden with a CIES utility function. This leads to a smooth decreasing consumption path and a positive research effort from sometime \tilde{t} (possibility $\tilde{t} = 0$ onwards). The sketch of the argument runs as follows (see Amigues-Grimaud-Moreaux (2003-a, section 6 and section 7) for the details). Let \bar{t} be the time at which the resource is exhausted in the present model ($\bar{t} = A_e^{-1}B_{eo}S_o$). Assume that the employment in the consumption good production sector is reduced from ℓ_{et} down to $\tilde{\ell}_{et} < 1$ over some short time interval $(\bar{t} - \Delta, \bar{t})$, with $\tilde{\ell}_{et}$ strictly decreasing, in order to lessen the consumption discontinuity gap. A first effect is that at any time $t, t \in (\bar{t} - \Delta, \bar{t})$, a quantity of resource $A_e B_{eo}^{-1}[1 - \tilde{\ell}_{et}]$ is saved, permitting a positive consumption over some time interval $(\bar{t}, \bar{t} + \Theta)$ (possibly $\Theta = +\infty$). But clearly the idle labor time interval $1 - \tilde{\ell}_{et}$ can be employed in the research sector over the time interval $(\bar{t} - \Delta, \bar{t})$ in order to improve B_e . Hence the amount of resource which can be saved is higher than $A_e B_{eo}^{-1} \int_{\bar{t} - \Delta}^{\bar{t}} (1 - \tilde{\ell}_{et}) dt$. Also the idle labor over the time interval $(\bar{t}, \bar{t} + \Theta)$ can be employed in the research sector too, to improve the efficient use of the saved resource. The argument may be used repeatedly as long as there exists a consumption discontinuity gap at \bar{t} . Thus, if the utility function is a CIES function, the consumption and research effort optimal paths are :

- either first to consume at the maximal rate A_e , i.e. $\ell_{et} = 1$ and $n_{et} = 0$, over some time interval $[0, t)$, and next decrease the employment in the consumption good sector and increase the employment in the research sector indefinitely, if either B_{eo} or S_o , or both, are sufficiently high ; - or drop the first phase during which $\ell_{et} = 1$ and from the start implement an increasing research effort policy, if either B_{eo} or S_o , or both, are sufficiently low.

- or drop the first phase during which $\ell_{et} = 1$ and from the start implement an increasing research effort policy, if either B_{eo} or S_o , or both, are sufficiently low.

4 The renewable resource economy

In the renewable resource economy there exists a critical level of scientific and technical knowledge capital, $\bar{B}_r = A_r \bar{r}^{-1}$, allowing all the available labor to be employed in the consumption good production sector. Thus if $B_{r_0} \geq \bar{B}_r$ either the knowledge capital or equivalently the flow \bar{r} of natural resource is abundant forever, and $\ell_{rt} = 1$ and $n_{rt} = 0, t \in [0, +\infty)$, is trivially the optimal policy. Hence what we have to determine is the optimal policy when initially the knowledge capital is lower than this critical level, i.e. when $B_{r_0} < \bar{B}_r$. We show that in this case the optimal research policy is the passive or residual policy, that is to employ in the research sector only this part of the labor supply which cannot be employed in the consumption good production sector, which part of the labor supply is positive as long as $B_{rt} < \bar{B}_r$. For such a policy the critical level \bar{B}_r is reached only at infinity¹³. This is the optimal policy whatever the productivity b_r in the research sector and the social rate of discount $\rho > 0$.

4.1 General characteristics of the optimal policies

Let LR_t be the Lagrangian of the program (PR) :

$$LR_t = A_r \ell_{rt} e^{-\rho t} + \nu_{rt} [\bar{r} - A_r B_{rt}^{-1} \ell_{rt}] + \mu_{rt} b_r n_{rt} B_{rt} + \omega_{rt} [1 - \ell_{rt} - n_{rt}] + \gamma_{r\ell,t} \ell_{rt} + \gamma_{rn,t} n_{rt}.$$

The first order conditions are :

$$\partial LR_t / \partial \ell_{rt} = 0 \Leftrightarrow A_r e^{-\rho t} - \nu_{rt} A_r B_{rt}^{-1} - \omega_{rt} + \gamma_{r\ell,t} = 0 \quad (4.1)$$

$$\partial LR_t / \partial n_{rt} = 0 \Leftrightarrow \mu_{rt} b_r B_{rt} - \omega_{rt} + \gamma_{rn,t} = 0 \quad (4.2)$$

together with the complementary slackness conditions :

$$\omega_{rt} \geq 0, 1 - \ell_{rt} - n_{rt} \geq 0, \text{ and } \omega_{rt} [1 - \ell_{rt} - n_{rt}] = 0 \quad (4.3)$$

$$\gamma_{r\ell,t} \geq 0, \ell_{rt} \geq 0, \text{ and } \gamma_{r\ell,t} \ell_{rt} = 0 \quad (4.4)$$

¹³With substitution possibilities between the inputs, it may be shown that there does not exist an optimal regular path for the renewable resource economy with a strictly positive R&D effort. Moreover in the long run the optimal R&D effort converges asymptotically to zero (see Amigues, Long, Moreaux, 2003)

$$\gamma_{rn,t} \geq 0, n_{rt} \geq 0, \text{ and } \gamma_{rn,t}n_{rt} = 0 \quad (4.5)$$

$$\nu_{rt} \geq 0, \bar{r} - A_r B_{rt}^{-1} \ell_{rt} \geq 0, \text{ and } \nu_{rt}[\bar{r} - A_r B_{rt}^{-1} \ell_{rt}] = 0 \quad (4.6)$$

The dynamics of the costate variable μ_{rt} is given by :

$$\dot{\mu}_{rt} = -\partial L R_t / \partial B_{rt} \Leftrightarrow \dot{\mu}_{rt} = -\nu_{rt} A_r B_{rt}^{-2} \ell_{rt} - \mu_{rt} b_r n_{rt} \quad (4.7)$$

Last the transversality condition at infinity is :

$$\lim_{t \uparrow +\infty} \mu_{rt} B_{rt} = 0 \quad (4.8)$$

As in the non renewable resource economy there exist two ways to decompose the discounted value of an optimal consumption plan, and the imputed value of the stock of knowledge $\mu_{rt} B_{rt}$ must be equal here to the capital value of the flow of the renewable resource $\bar{r} \int_t^\infty \nu_{r\tau} d\tau$.

Since by (4.4) $\gamma_{r\ell,t} \ell_{rt} = 0$, then multiplying (4.1) by ℓ_{rt} , we get :

$$A_r \ell_{rt} e^{-\rho t} - \nu_{rt} A_r B_{rt}^{-1} \ell_{rt} - \omega_{rt} \ell_{rt} = 0$$

that is :

$$c_t e^{-\rho t} = \nu_{rt} A_r B_{rt}^{-1} \ell_{rt} + \omega_{rt} \ell_{rt}$$

By (4.6) $\nu_{rt} A_r B_{rt}^{-1} \ell_{rt} = \nu_{rt} \bar{r}$, hence :

$$c_t e^{-\rho t} = \nu_{rt} \bar{r} + \omega_{rt} \ell_{rt},$$

and :

$$W_t(C) = \bar{r} \int_t^\infty \nu_{r\tau} d\tau + \int_t^\infty \omega_{r\tau} \ell_{r\tau} d\tau, \quad t \geq 0.$$

Now multiplying (4.7) by B_{rt} and remembering that $b_r n_{rt} B_{rt} = \dot{B}_{rt}$ we get, taking (4.6) into account :

$$(\mu_{rt} \dot{B}_{rt}) = -\nu_{rt} A_r B_{rt}^{-1} = -\nu_{rt} \bar{r}.$$

Integrating from t up to $+\infty$ and noting that $\mu_{r\infty} B_{r\infty} = 0$ by (4.8), we obtain :

$$\mu_{rt} B_{rt} = \bar{r} \int_t^\infty \nu_{r\tau} d\tau, \quad t \geq 0, \quad (4.9)$$

hence :

$$W_t(C) = \mu_{rt}B_{rt} + \int_t^\infty \omega_{r\tau}l_{r\tau}d\tau, t \geq 0$$

Proposition 5 In the renewable resource economy, along the optimal path, at each point of time, the capital value of the flow \bar{r} of the renewable resource must be equal to the imputed value of the technological knowledge stock and the discounted value of the optimal consumption plan C is given by :

$$W_t(C) = \bar{r} \int_t^\infty \nu_{r\tau}d\tau + \int_t^\infty \omega_{r\tau}l_{r\tau}d\tau = \mu_{rt}B_{rt} + \int_t^\infty \omega_{r\tau}l_{r\tau}d\tau, t \geq 0.$$

4.2 Optimal policy

Since the instantaneous utility is a linear function of the instantaneous consumption rate one would perhaps be tempted to conclude that the optimal policy is to go as fast as possible to the critical knowledge capital level \bar{B}_r , provided that the productivity in the research sector be sufficiently high relative to the social rate of discount. As we shall show, this is not the good intuition.

Let us define an active research policy as any policy $\{(l_{rt}, n_{rt}), t \geq 0\}$ implying that there exists some time interval $(t_1, t_2), 0 \leq t_1 < t_2$, during which $B_{rt} < \bar{B}_r$ and the employment in the research sector is so high that the resource constraint (PR.1), $\bar{r} - A_r B_{rt}^{-1}(1 - n_{rt}) \geq 0$, is not binding. The activist research policy is this policy consisting in going as soon as possible to the critical level \bar{B}_r , that is $n_{rt} = 1$ as long as $B_{rt} < \bar{B}_r$. The passive research policy is the policy consisting in first consuming, and searching iff all the available labor cannot be employed in the consumption good production sector, that is, if $B_{rt} < \bar{B}_r$, then $n_{rt} = 1 - A_r^{-1}B_{rt}\bar{r}$.¹⁴

Let us first show that any active policy is non optimal in the present context.

¹⁴Note that such a clearcut distinction between an active and a passive research policy is only meaningful under the Leontief technology assumption. With inputs substitutability it may be shown that it is always optimal to consume all the available resource flow. But in this case, the use of labor in production is depending upon the ratio between the inputs efficiencies. This ratio, which is equal to one in the Leontief case, will be endogeneously determined along a optimal path (see Amigues, Long, Moreaux, 2003).

4.2.1 Non optimality of the active research policies

Let us first consider the activist policy $\ell_{rt} = 0$ and $n_{rt} = 1$ as long as $B_{rt} < \bar{B}_r$, so that :

$$B_{rt} < \bar{B}_r \Rightarrow B_{rt} = B_{r0}e^{b_r t}$$

The critical level \bar{B}_r is attained at this time T at which $B_{rt} = \bar{B}_r$, that is :

$$T = b_r^{-1}[\log \bar{B}_r - \log B_{r0}].$$

Now let us examine the following alternative policy. The society reduces its research effort over the time interval $[0, t_1]$ by dn and next go back to $n_{rt} = 1$ up to this time at which $B_{rt} = \bar{B}_r$. B_{rt} is now given by :

$$B_{rt} = \begin{cases} B_{r0}e^{b_r(1-dn)t} & , t \in [0, t_1) \\ B_{r0}e^{b_r(1-dn)t_1}e^{b_r(t-t_1)} & , t \in [t_1, T + dT) \\ \bar{B}_r & , t \in [T + dT, +\infty) \end{cases}$$

where $dT = t_1 dn$.

In terms of welfare, the balance sheet of the consumption variations induced by this policy change can be drawn up as follows :

i - over the time interval $[0, t_1]$ the consumption rate is now higher, $A_r dn$ instead of 0, thus the discounted gain is equal to :

$$\int_0^{t_1} A_r dn e^{-\rho t} dt = \rho^{-1} A_r dn [1 - e^{-\rho t_1}];$$

ii - over the time interval $[t_1, T)$ the consumption rate does not change, that is $c_t = 0$;

iii - over the time interval $[T, T + dT)$ the consumption rate shuts down to 0 from the A_r level, hence the discounted loss is equal to :

$$\int_T^{T+dT} A_r e^{-\rho t} dt = \rho^{-1} A_r e^{-\rho T} [1 - e^{-\rho dT}] dn;$$

iv - last over the time interval $[T + dT, +\infty)$ the consumption rate does not change.

Summing up, the welfare variation ΔW_{t_1} is equal to :

$$\Delta W_{t_1} = \rho^{-1} A_r dn \{ [1 - e^{-\rho t_1}] - e^{-\rho T} [1 - e^{-\rho t_1 dn}] \},$$

so that :

$$dW_{t_1}/dt_1 = A_r dn [e^{-\rho t_1} - dne^{-\rho [T+t_1 dn]}]$$

and at $t_1 = 0$:

$$(dW_{t_1}/dt_1) |_{t_1=0} = A_r dn [1 - dne^{-\rho T}]$$

Hence, for any $dn \in (0, 1]$, we have :

$$(dW_{t_1}/dt_1) |_{t_1=0} > 0$$

We conclude that over the time interval $[0, t_1)$ reducing the employment allocated to the research sector for increasing the employment in the consumption good production sector, is better than not. Since $B_{rt} < \bar{B}_r$, $t \in [0, t_1)$ there is a ceiling on the employment in the consumption good sector, and we must have $n_{rt} = 1 - A_r^{-1} B_{rt} \bar{r}$.

Now let us remark that in the preceding argument the time $t = 0$ and the value of $B_{r0} < \bar{B}_r$ are both arbitrary. Thus it happens that not only the activist policy cannot be optimal but also any active policy and we are left with the passive policy as the only candidate. What has to be emphasized is that the conclusion holds whatever the productivity of the research effort and the social rate of discount.

4.2.2 The passive research policy

If the resource constraint (PR.1) is tight then :

$$\bar{r} = A_r B_{rt}^{-1} \ell_{rt} \Rightarrow \ell_{rt} = A_r^{-1} B_{rt} \bar{r} \text{ and } n_{rt} = 1 - A_r^{-1} B_{rt} \bar{r},$$

hence :

$$\dot{B}_{rt} = b_r [1 - A_r^{-1} B_{rt} \bar{r}] B_{rt}.$$

The solution of this differential equation is the following logistic function :

$$B_{rt} = \{ \bar{r} A_r^{-1} + [B_{r0}^{-1} - \bar{r} A_r^{-1}] e^{-b_r t} \}^{-1} = \{ \bar{B}_r^{-1} + [B_{r0}^{-1} - \bar{B}_r^{-1}] e^{-b_r t} \}^{-1}. \quad (4.10)$$

Thus :

$$\dot{B}_{rt} = 0 \Leftrightarrow B_{rt} = A_r \bar{r}^{-1} = \bar{B}_r,$$

and since $B_{r0} < \bar{B}_r$, the critical level of the technical knowledge capital \bar{B}_r is attained only asymptotically.

The instantaneous consumption rate is always increasing towards its maximal level A_r :

$$c_t = B_{rt}\bar{r}, \dot{c}_t = \dot{B}_{rt}\bar{r} > 0, t \in [0, +\infty), \text{ and } \lim_{t \uparrow +\infty} c_t = A_r.$$

We show in Appendix (§6.2.2) that this policy is truly the optimal one.

Proposition 6 *In the renewable resource economy whatever the productivity of the research effort b_e and the social rate of discount ρ , the passive policy is the only optimal policy.*

The time profile of the resource productivity index B_{et} is illustrated in Figure 2 below. If B_{e0} is low, lower than $\bar{B}_r/2$, $B_{e\hat{t}} = \bar{B}_1/2$, and next increasing at a decreasing rate. If then B_{rt} is first increasing at an increasing rate up to than time \hat{t} at which B_{e0} is low, lower than $\bar{B}_r/2$, then B_{et} is always increasing at a decreasing rate.

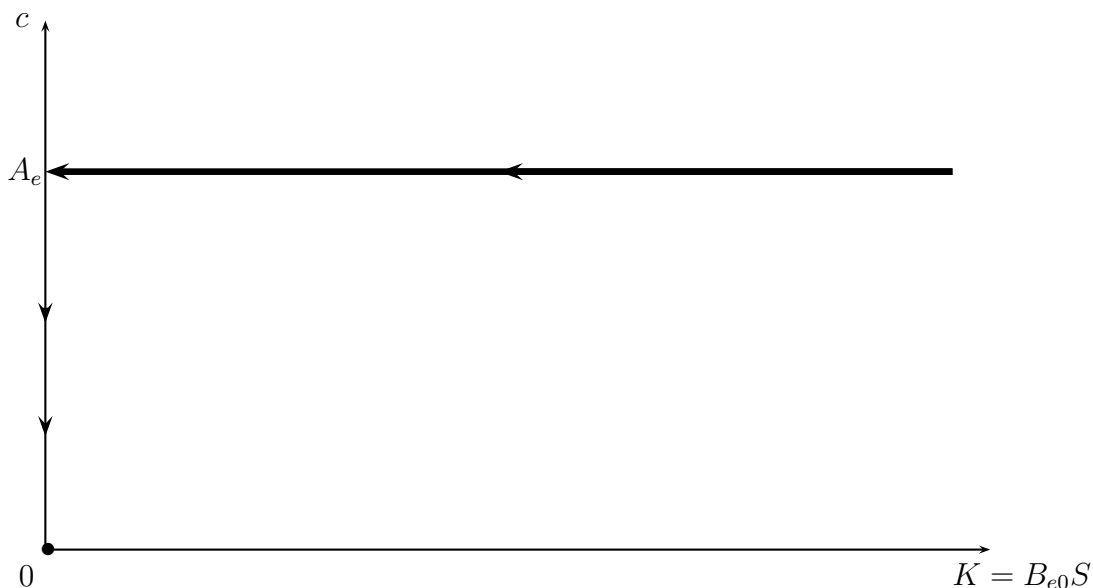


Figure 2 - Optimal path in the non renewable resource economy. Case $b_e < \rho$

5 Conclusion

It appears that the optimal R& D policies are very different in the pure renewable resource economy and in the pure non renewable resource economy. Such a difference doesn't arise from value repartition considerations. In both cases, the standard accounting condition for multisectoral optimal growth models applies, stating that along any optimal consumption path, welfare has to be divided between the discounted sum of the productive labor value flow and either the imputed value of the exhaustible resource stock in the non renewable economy case, either the discounted sum of the renewable resource value flow in the renewable economy case (see Propositions 1 and 5).

For the non renewable economy case, we have shown that an active research policy is always optimal when the consumption potential of the economy is initially below its sustainable long run level, provided that the productivity of the research effort is higher than the social rate of impatience. In our model framework, this implies that the interest of the society is to catch up in finite time the sustainable long run optimum.

In the renewable resource economy, such an active research policy is never optimal for any levels of the research productivity parameter and the social discount rate. To the contrary the society has to consume all the available renewable resource flow forever and to devote only the residual fraction of its labor force not used in production in order to improve the productive efficiency factor of the natural resource. As a consequence the sustainable long run optimum is only reached asymptotically.

Hence the analysis suggests that the typically slow pattern of technical progress observed in the so-called "organic societies" may not only result from the lack of technical skills or scientific knowledge in such societies but also from the optimal conditions for their historical development.

A first question raised by our results is whether they are robust to the introduction of substitution possibilities between the inputs. In a work in progress (Amigues, Long, Moreaux, 2003) we have already shown that the qualitative results obtained in the non renewable economy case translate in an economy with a CIES utility function and a CES production technology. Under the same set of assumptions, the renewable resource economy doesn't converge to some optimal regular path with a strictly positive R&D effort. This effort must go to zero asymptotically.

Clearly a next step for this research is to determine what are the optimal policies in economies in which both types of resources are available and, like

in the present ones, the specific productivity of the different kinds of resources can be improved through time¹⁵.

6 Appendix

6.1 The non renewable resource economy

We show that for the policies defined in Propositions 3 and 4 there exist non negative continuous time functions μ_{et} and ω_{et} and non negative time functions $\gamma_{el,t}$ and $\gamma_{en,t}$ such that for some $\lambda > 0$ all the conditions (3.1) to (3.8) are satisfied.

6.1.1 Case $b_e > \rho$ and $K_0 > \bar{K}$

In this case $\ell_{et} = 1$ and $n_{et} = 0$ over a first time interval $[0, T)$, so that for $t \in [0, T)$:

$$B_{et} = B_{e0}, s_t = A_e B_{e0}^{-1} \equiv s, S_t = S_0 - A_e B_{e0}^{-1} t, \text{ and } K_t = B_{e0} S_0 - A_e t. \quad (6.1)$$

From $K_T = \bar{K}$, we get :

$$T = A_e^{-1} B_{e0} S_0 - (b_e - \rho)(\rho b_e)^{-1}, \text{ and } S_T = A_e B_{e0}^{-1} (b_e - \rho)(\rho b_e)^{-1}.$$

Note that S_t may equivalently be expressed as :

$$S_t = S_T + s[T - t] = S_T + A_e B_{e0}^{-1} [T - t]. \quad (6.2)$$

Now from (3.19) for $B_{et} = B_{e0}$, we obtain the value of λ :

$$\lambda = \rho b_e^{-1} B_{e0} e^{-\rho T} \quad (6.3)$$

Next, (3.9) for S_t given by (6.2) and $B_{et} = B_{e0}$, results in :

$$\mu_{et} = \lambda [B_{e0}^{-1} S_T + A_e B_{e0}^{-2} [T - t]], \quad (6.4)$$

hence :

$$\dot{\mu}_{et} = -\lambda A_e B_{e0}^{-2},$$

¹⁵For a first attempt in characterizing the optimal consumption in economies in which both resources are available and the technical progress affecting the productivity is exogenous, see Amigues, Long and Moreaux (2002).

that is the value of μ_{et} given by (3.7) for $\ell_{et} = 1$ and $n_{et} = 0$. Note also that, for $t = T$, (6.3) \cup (6.4) implies that :

$$\mu_{et} = \rho b_e^{-1} S_T e^{-\rho T}$$

that is μ_{et} as given by (3.19) for $t = T$. Thus μ_{et} is continuous at $t = T$.

Let us now consider (3.1) for $B_{et} = B_{e0}$ and $\gamma_{el,t} = 0$:

$$A_e e^{-\rho t} - \lambda A_e B_{e0}^{-1} = \omega_{et}.$$

Since $\gamma_{en,t} \geq 0$, we get from (3.2) :

$$\omega_{et} \geq b_e \mu_{et} B_{et}.$$

But $\mu_{et} B_{et} = \lambda S_t$ by (3.9) and $B_{et} = B_{e0}$ for $t \in [0, T]$, hence it is sufficient to show that :

$$A_e B_{e0} e^{-\rho t} \geq \lambda [b_e B_{e0} S_t + A_e]$$

Substituting for λ its value given by (6.3) and for S_t its value given by (6.2), the above inequality results in :

$$(b_e e^{\rho(T-t)} - \rho)(\rho b_e)^{-1} \geq (b_e - \rho)(\rho b_e)^{-1} + [T - t].$$

Clearly this weak inequality is satisfied as an equality for $t = T$ and as a strong inequality for $t \in [0, T)$ if $b_e > \rho$.

6.1.2 Case $b_e > \rho$ and $K_0 < \bar{K}$

Here $\ell_{et} = 0$ and $n_{et} = 1$ over a first time interval $[0, T)$ so that for $t \in [0, T)$:

$$B_{et} = B_{e0} e^{b_e t}, S_t = S_0, \text{ and } K_t = B_{e0} S_0 e^{b_e t}.$$

From $K_T = \bar{K}$, we get :

$$e^{-b_e T} = \rho b_e [A_e (b_e - \rho)]^{-1} B_{e0} S_0, \quad (6.5)$$

that is :

$$T = b_e^{-1} \{ \log [A_e (b_e - \rho)] - \log [\rho b_e B_{e0} S_0] \}.$$

The value of λ is obtained from (3.9) for $B_{et} = B_{e0} e^{b_e T}$:

$$\lambda = \rho b_e^{-1} B_{e0} e^{(b_e - \rho) T}$$

Now, concerning μ_{et} , in (3.9) let us substitute for λ its above value, for S_t, S_0 and for $B_{et}, B_{e0}e^{b_e t}$, so that (3.9) results in :

$$\mu_{et} = \rho b_e^{-1} S_0 e^{b_e(T-t)} e^{-\rho T},$$

hence :

$$\dot{\mu}_{et} = -b_e \mu_{et},$$

that is the value of $\dot{\mu}_{et}$ given by (3.7) for $\ell_{et} = 0$ and $n_{et} = 1$. Furthermore for $t = T$, we have :

$$\mu_{et} = \rho b_e^{-1} S_0 e^{-\rho T},$$

that is μ_{et} given by (3.19) for $t = T$ and $S_t = S_0$. Thus μ_{et} is continuous at $t = T$.

Let us now consider (3.1) for $\gamma_{e\ell,t} \geq 0$:

$$A_e e^{-\rho t} - \lambda A_e B_{et}^{-1} \leq \omega_{et}.$$

Since $\gamma_{en,t} = 0$, we get from (3.2) :

$$b_e \mu_{et} B_{et} = \omega_{et}$$

hence, remembering that $\mu_{et} B_{et} = \lambda S_t$, it is sufficient to show that :

$$A_e e^{-\rho t} - \lambda A_e b_e^{-1} \leq \lambda b_e S_{et}.$$

Multiplying the both sides of the above inequality by B_{et} and substituting for λ, B_{et} and S_t their above expressions, we get :

$$(\rho b_e)^{-1} A_e [b_e e^{-b_e T} e^{-\rho(T-t)} - \rho e^{-b_e t}] \leq b_{e0} S_0.$$

In the above inequality let us substitute for $e^{-\rho T}$ its value given by (6.5).

We obtain :

$$B_{e0} e^{b_e t} S_0 \leq A_e (b_e - \rho) \{ [b_e (e^{\rho(t-T)} - 1) + \rho] - e^{b_e t} \}^{-1}.$$

For $t = T$, the both sides of this weak inequality take the same value \bar{K} . Over the time interval $[0, T)$, the left hand side is increasing whereas the right hand side is decreasing since $b_e - \rho > 0$. Hence for $t \in [0, T)$, the above inequality is satisfied as a strong inequality.

6.1.3 Case $b_e < \rho$.

We want to show that in this case the optimal policy is to consume at the maximal rate up to this time T at which the resource stock is exhausted, that is $\ell_{et} = 1$ and $n_{et} = 0, t \in [0, T)$, so that s_t, S_t and K_t are given by (6.1). Again S_t is equivalently given by (6.2).

From $S_T = 0$ we get :

$$T = A_e^{-1} B_{e0} S_0.$$

The value of λ is the marginal value of a small additional stock of resource. The increase in consumption generated by an additional quantity of resource dS , available at time $t = T$, is equal to A_e at each point of time over an interval $[T, T + dT)$, $dT = A_e^{-1} B_{e0} dS$. Thus the increase in welfare is equal to :

$$dW = A_e e^{-\rho T} dT = B_{e0} e^{-\rho T} dS,$$

so that :

$$dW/dS = B_{e0} e^{-\rho T} \Rightarrow \lambda = B_{e0} e^{-\rho T} \quad (6.6)$$

Next from (3.9), for λ given by (6.6), S_t given by (6.2) with $S_T = 0, B_{et} = B_{e0}, t \in [0, T)$, and $S_t = 0, t \in [T, +\infty)$, we get :

$$\mu_{et} = \begin{cases} A_e B_{e0}^{-1} e^{-\rho T} [T - t] & , t \in [0, T) \\ 0 & , t \in [T, +\infty) \end{cases}$$

Clearly μ_{et} is continuous at $t = T$ and :

$$\dot{\mu}_{et} = \begin{cases} -A_e B_{e0}^{-1} e^{-\rho T} & , t \in [0, T) \\ 0 & , t \in [T, +\infty) \end{cases}$$

Hence, for $t \in [0, T)$, (3.7) is satisfied with $\ell_{et} = 1, n_{et} = 0$ and λ given by (6.6), whereas for $t \in [T, +\infty)$, (3.7) is also satisfied, with $\ell_{et} = 0$ and $n_{et} = 0$.

Equation(3.1) with $\gamma_{el,t} = 0$ and λ given by (6.6) results in :

$$\omega_{et} = A_e [e^{-\rho t} - e^{-\rho T}]$$

and (3.2) with $\gamma_{en,t} \geq 0$, in :

$$b_e \mu_{et} B_{et} \leq \omega_{et},$$

that is, taking (3.9) into account with λ given by (6.6) :

$$b_e A_e e^{-\rho T} [T - t] \leq \omega_{et}.$$

Thus it is sufficient to show that :

$$b_e e^{-\rho T} [T - t] \leq e^{-\rho t} - e^{-\rho T}.$$

Clearly for $t = T$ the above weak inequality is satisfied as an equality and since $b_e < \rho$, as a strong inequality for $t \in [0, T)$.

Last for $t \in [T, +\infty)$, we have :

$$\omega_{et} = 0, \gamma_{el,t} = -A_e [e^{-\rho t} - e^{-\rho T}], \text{ and } \gamma_{en,t} = -b_e A_e e^{-\rho T} [T - t].$$

6.2 The renewable resource economy

6.2.1 Preliminary remarks

For B_{rt} given by (4.10) we have, for any $t \geq 0$ and any $\tau \geq t$:

$$B_{r\tau} = \bar{B}_r \{1 + [\bar{B}_r B_{rt}^{-1} - 1] e^{-b_r(\tau-t)}\}^{-1}. \quad (6.7)$$

Differentiating this expression with respect to B_{rt} , we get :

$$\partial B_{r\tau} / \partial B_{rt} = B_{r\tau}^2 B_{rt}^{-2} e^{-b_r(\tau-t)}. \quad (6.8)$$

Now, let us differentiate (6.7) with respect to time t :

$$\frac{\partial B_{r\tau}}{\partial t} = - \frac{\bar{B}_r [-\bar{B}_r \dot{B}_{rt} B_{rt}^{-2} + b_r (\bar{B}_r B_{rt}^{-1} - 1)] e^{-b_r(\tau-t)}}{[1 + [\bar{B}_r B_{rt}^{-1} - 1] e^{-b_r(\tau-t)}]^2}$$

Substituting for \dot{B}_{rt} its value $b_r n_{rt} B_{rt}$, we get :

$$\frac{\partial B_{r\tau}}{\partial t} = - \frac{b_r \bar{B}_r [\bar{B}_r B_{rt}^{-1} \ell_{rt} - 1] e^{-b_r(\tau-t)}}{[1 + [\bar{B}_r B_{rt}^{-1} - 1] e^{-b_r(\tau-t)}]^2}$$

and since $\ell_{rt} = \bar{B}_r^{-1} B_{rt}$ the above expression results in :

$$\partial B_{r\tau} / \partial t = 0. \quad (6.9)$$

6.2.2 Proof of Proposition 6

Consider the value at time t of the objective function (PR.0) under the passive investment policy, that is $W_t e^{\rho t}$, denoted by V_t :

$$V_t = \int_t^\infty \bar{r} B_{r\tau} e^{-\rho(\tau-t)} d\tau, \quad (6.10)$$

where $B_{r\tau}$ is given by (6.7).

If the passive investment is to be the optimal policy, we must have :

$$\mu_{rt} = e^{-\rho t} \frac{\partial V_t}{\partial B_{rt}}$$

From (6.8) and (6.10), we get :

$$\frac{\partial V_t}{\partial B_{rt}} = \bar{r} \int_t^\infty B_{r\tau}^2 B_{rt}^{-2} e^{-(b_r+\rho)(\tau-t)} d\tau,$$

hence :

$$\mu_{rt} = \bar{r} e^{-\rho t} \int_t^\infty B_{r\tau}^2 B_{rt}^{-2} e^{-(b_r+\rho)(\tau-t)} d\tau = \bar{r} e^{b_r t} B_{rt}^{-2} \int_t^\infty B_{r\tau}^2 e^{-(b_r+\rho)\tau} d\tau$$

Since $\lim_{t \uparrow +\infty} B_{rt} = \bar{B}_r$, the transversality condition (4.8) is satisfied iff $\lim_{t \uparrow +\infty} \mu_{rt} = 0$. Note that $B_{r0} \leq B_{rt} \leq B_{r\tau} < \bar{B}_r$ for any $0 \leq t \leq \tau$, so that $B_{r\tau} B_{rt}^{-1}$ is bounded from above by $\bar{B}_r B_{r0}^{-1} > 1$. Hence :

$$\int_t^\infty B_{r\tau}^2 B_{rt}^{-2} e^{-(b_r+\rho)(\tau-t)} d\tau \leq (b_r + \rho)^{-1} (\bar{B}_r B_{r0}^{-1})^2,$$

and :

$$\lim_{t \uparrow +\infty} \mu_{rt} \leq \bar{r} (b_r + \rho)^{-1} (\bar{B}_r B_{r0}^{-1})^2 \lim_{t \uparrow +\infty} e^{-\rho t} = 0.$$

Now since $\gamma_{rn,t} = 0$, (4.2) is satisfied iff $w_{rt} = b_r \mu_{rt} B_t$ which is positive for any $t \geq 0$.

Next let us turn to the condition (4.7). Multiplying the both sides by B_{rt} , we get :

$$(\mu_{rt} \cdot B_{rt}) = -\nu_{rt} \bar{r}$$

Let us introduce the following notation :

$$J_t \equiv B_{rt}^{-1} e^{b_r t} \int_t^\infty B_{r\tau}^2 e^{-(b_r+\rho)\tau} d\tau,$$

so that :

$$\mu_{rt}B_{rt} = \bar{r}J_t \text{ and } \nu_{rt} = -\dot{J}_t.$$

Since $\partial B_{r\tau}/\partial t = 0$ and $\dot{B}_{rt} = b_r n_{rt} B_{rt}$, then :

$$\dot{J}_t = -B_{rt}e^{-\rho t} + [1 - n_{rt}]b_r B_{rt}^{-1} e^{b_r t} \int_t^\infty B_{r\tau}^2 e^{-(b_r+\rho)\tau} d\tau. \quad (6.11)$$

Taking into account that $1 - n_{rt} = \ell_{rt}$ and $A_r \ell_{rt} = B_{rt} \bar{r} \Rightarrow \ell_{rt} B_{rt}^{-1} = \bar{B}_r^{-1}$, we get finally :

$$\nu_{rt} = -\dot{J}_t = B_{rt}e^{-\rho t} - b_r \bar{B}_r^{-1} e^{b_r t} \int_t^\infty B_{r\tau}^2 e^{-(b_r+\rho)\tau} d\tau.$$

It remains to show that $\nu_{rt} \geq 0$, that is, using (6.11) and after some simple manipulations, that :

$$B_{rt} \geq b_r \int_t^\infty B_{r\tau} e^{-(b_r+\rho)(\tau-t)} d\tau - \int_t^\infty \dot{B}_{r\tau} e^{-(b_r+\rho)(\tau-t)} d\tau.$$

Integrating by parts the second integral of the right hand side, we get :

$$-B_{rt} + (b_r + \rho) \int_t^\infty B_{r\tau} e^{-(b_r+\rho)(\tau-t)} d\tau.$$

Hence $\nu_{rt} \geq 0$ is equivalent to :

$$0 \geq -\rho \int_t^\infty B_{r\tau} e^{-(b_r+\rho)(\tau-t)} d\tau,$$

which is clearly satisfied.

By construction of the above values of μ_{rt} and ν_{rt} , (4.7) is satisfied.

Last we have to check that for the above values of ω_{rt} and ν_{rt} and for $\gamma_{r\ell,t} = 0$, (4.1) is satisfied. For these values the left hand side of (4.1) takes the following value :

$$A_r e^{-\rho t} - [B_{rt} e^{-\rho t} - \omega_{rt} \ell_{rt} \bar{r}^{-1}] A_r B_{rt}^{-1} - \omega_{rt} = \omega_{rt} \ell_{rt} A_r B_{rt}^{-1} \bar{r}^{-1} - \omega_{rt}.$$

Since $A_r \ell_{rt} = B_{rt} \bar{r}$ the right hand side of the above equality is equal to 0 so that (4.1) is satisfied.

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