C.I.R.P.É.E
Centre Interuniversitaire sur le Risque,
les Politiques Économiques et l'Emploi

Cahier de recherche/Working Paper **07-10**

Bond Indebtedness in a Recursive Dynamic CGE Model

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Mai/May 2007

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An earlier, incomplete version of this working paper was published in French in 2005 as CIRPÉE working paper 05-05, under the title « La dette obligataire dans un MÉGC dynamique séquentiel ».

Abstract :

In this paper, we present a minimalist version of a model of bond financing and debt, imbedded in a stepwise dynamic CGE model. The proposed specification takes into account the main characteristics of bond financing. Bonds compete on the securities market with shares, so that the yield demanded by the buyers of new bond issues increases as the cumulative bond debt grows relative to the stock of outstanding shares.
Restrictions are imposed on the maturity structure of bonds, so that it is possible to attain a reasonable compromise between a realistic representation of the evolution of the debt, and the demands on model memory of past variables values which impinge on the current period.
In the proposed model, the borrowing government reimburses bonds that have reached maturity, and pays interest on the outstanding debt. The prices of bonds issued at different periods and with different maturities are consistent with an arbitrage equilibrium. The supply of new bonds and of new shares is determined by the government's and business's borrowing needs. Security demand reflects the rational choices of portfolio managing households, following a version of the Decaluwé-Souissi model.
These notions are illustrated with fictitious data in model EXTER-Debt. The full specification of the model is described, and simulation results are presented which demonstrate model properties.

Key words : CGE models; recursive dynamics; bond debt; financial assets

JEL codes : C68, D58, G1, H63

Résumé:

Dans ce texte, nous présentons une version minimaliste d'un modèle de la dette obligataire qui s'inscrit dans un modèle d'équilibre général calculable dynamique séquentiel. La spécification proposée tient compte des principales caractéristiques suivantes des obligations. Les obligations sont en concurrence avec une autre catégorie d'actifs, les actions, de sorte que le rendement exigé par les acheteurs de nouvelles obligations augmente à mesure qu'augmente la dette obligataire par rapport au stock d'actions en circulation. Des restrictions imposées à la structure de maturité des obligations permettent de définir un compromis raisonnable entre le réalisme de la représentation de l'évolution de la dette obligataire et le poids des valeurs passées des variables que le modèle doit conserver en mémoire.
Dans le modèle proposé, l'État emprunteur rembourse les obligations arrivées à échéance et paie les intérêts sur la dette en cours. Les prix des obligations émises à différents moments avec des échéances différentes sont cohérents avec un équilibre d'arbitrage. L'offre de nouvelles obligations et actions est déterminée par les besoins d'emprunt de l'État et des entreprises. La demande d'actifs reflète le comportement rationnel des ménages gestionnaires de portefeuille, conformément au modèle Decaluwé-Souissi.
Cette conception est illustrée au moyen de données fictives dans le modèle EXTER-Debt. La spécification complète du modèle est donnée et des résultats de simulations en montrent les propriétés.

Mots Clés: Modèles d'équilibre général concurrentiel; dynamique séquentielle; dette obligataire; actifs financiers

Classification JEL: C68, D58, G1, H63

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INTRODUCTION

Bond indebtedness...

This article is about bond indebtedness in a recursive dynamic computable general equilibrium model. The objective of the proposed specification is to take into account the following characteristics of bonds :

- they are issued at a given date;
- they have a given nominal, or face value;
- they bear interest at a given rate relative to their face value;
- they have a maturity date, at which they are reimbursed by the issuer to the holder.

The policy interest of modeling bond indebtedness as accurately as possible is clear : any issuer of securities, including the government, runs the risk, beyond a certain level of indebtedness, that his/her credit rating fall, which then forces new issues to bear interest at increased rates, and may even close the door to further borrowing.

in a recursive dynamic CGE model

A recursive dynamic model is different from an intertemporal model in that, in the latter, the optimizing behavior of economic agents encompasses all periods up to the time horizon simultaneously, while in recursive dynamics, the system's state in each period depends only on past and current values of variables, not on future values (though it may depend on *expected* future values).

In principle, the model's « memory » at any given time could reach back to every preceding period, beginning with the initial situation. But if a simulation extends over more than very few periods, the number of variables quickly becomes quite large. That is why, practically speaking, modeling strategies try to condense memory of the past into a small number of stock variables. For instance, the stock of capital at the beginning of a period typically summarizes all of the previous history of investment and depreciation (except in vintage models, where the number of generations is nonetheless usually kept low).

With regard to debt in the form of bonds, there are three aspects which call into play the model's « memory ». First, the amount of interest payable depends on the face values and the interest rates of all past issues which have not yet been redeemed. Second, the amount of debt that comes to maturity (and must be redeemed or refinanced) depends on the face values and maturity dates of all past issues still outstanding. Finally, the level of indebtedness is the result of past issues, that is, in the case of government, of the cumulative deficit of past government expenditures and investment spending.

In this essay, we propose a model which can account for interest payments, debt redemption at maturity, and the level of indebtedness, while maintaining acceptable model memory requirements.

Our model does not pretend to be immediately operationnal. Our objective is rather to present the general principle of the proposed specification in a minimalist form. It is a model without money, where only relative prices matter. Although it has asset markets, it cannot be considered to be a financial model. Moreover, there is no financial intermediation in the model.

A minimalist framework

Even a minimalist framework requires that there be an asset competing with bonds. Indeed, in order to represent the rise in the cost of borrowing and the erosion of borrowing capacity which results from higher indebtedness, the rate of interest on new issues must depend on the stock of debt. In a micro-founded approach, that requires competition to government bonds from of at least one other asset. When government bonds compete with another asset, the greater the stock of outstanding debt, the lower the market valuation of bonds, and the higher the interest rate on new government bond issues.

That modeling strategy also implies that the demand for assets reflect the portfolio allocation behavior of asset holders. Moreover, not only current savings, but all of the wealth portfolio must be reallocated in every period. Because, if only current savings are allocated among currently offered new assets, equilibrium prices of new issues are independent of outstanding stocks[1].

These notions are illustrated in the EXTER-Debt model, which was developed to demonstrate the feasibility of the modeling principle presented in this paper. It was constructed on the basis of the EXTER_DS[2] model, which itself derives from the EXTER[3] model.

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1 The latter approach is Robinson's (1991) or Decaluwé, Martin and Souissi's (1992) « flow of funds » approach.

2 Forthcoming.

3 Decaluwe, Martens, Savard, 2001. Details of the EXTER model are available on the *Poverty and Economic Policy* (PEP) website (http://www.pep-net.org/) : see the « Core Training Manuals on CGE Modeling » section in the MPIA training material, *Volume 2 – Basic CGE models (training material)*.

In EXTER-Debt, we make several assumptions to make the model as simple as possible, even at some cost in terms of realism, so as not to distract from the illustration of the proposed modeling approach.

Our model has four agents : households, businesses, government, and the Rest-of-the-World (RoW) :

- Government issues bonds to finance its current deficit and public investment.
- Businesses issue shares to finance their investment expenditures.
- Households own a portfolio of both assets.
- The RoW owns only shares (one of our many simplifying assumptions).

Time structure of EXTER-Debt

We shall see below that financial asset markets are to be represented as transactions on stocks of assets. So it is of utmost importance to clearly specify the meaning of the time subscripts attached to stock variables. It is a matter of convention, of course, but it is preferable that the convention be explicit.

In this model, agents (households, businesses, government) make their decisions at the beginning of each period in the form of strategies (such as demand or supply functions), according to equilibrium values which they take as given (as is the case with prices in perfect competition). The equilibrium conditions determine end-of-period values.

Securities issued during a given period t start paying income (interest or dividends) in the following $t + 1$ period. Securities redeemed during a given t period bear interest in that same final period. But securities redeemed in period t are excluded from the stock of outstanding securities offered to portfolio owners during period t.

Organization of the paper

The following section defines bond price calculations, and develops the calculation formulas for the amount of bonds to be redeemed in the current period, interest to be paid on outstanding bonds, and the aggregate market value of bonds issued in previous periods not currently coming to maturity. The next section describes the Decaluwé-Souissi portfolio allocation model applied to households. Section 3 details asset market equilibrium conditions. In section 4, after a presentation of the Jung-Thorbecke investment demand function, closely related to Tobin's « *q* » theory, the key role of the user-cost of capital and its relationship to the asset market are examined. Simulation experiments, conducted on the basis of artificial data, are analyzed in the following section. The paper ends with a summary and conclusions.

1. BOND MARKET

1.1 The price of bonds

Recall that securities issued during a given period start paying income (interest or dividends) in the following period. Securities redeemed during a given period bear interest in that final period. But redeemed securities are excluded from the stock of outstanding securities offered to portfolio owners during that period.

A bond issued at face value in period t, maturing in period t+1, and bearing interest at current rate i_t^B, has a cost of acquisition in period t of 1, and a capitalized value in t+1 of $(1+i_t^B)$. After the payment of interest due in period t, a bond issued in period $t-\theta$, coming to maturity in period t+1, and bearing interest at rate $i_{t-\theta}^B$, has a capitalized value in t+1 of $(1+i_{t-\theta}^B)$. Assuming there is no risk aversion, in equilibrium, a portfolio manager must be indifferent between these two securities. Their relative price in period t is therefore

$$\frac{1+i_{t-\theta}^B}{1+i_t^B}$$

Next, a bond issued in period t at face value, coming to maturity in period $t+\tau$, and bearing interest at current rate i_t^B, has a cost of acquisition in t of 1, and a capitalized value in $t+\tau$ of $(1+i_t^B)^\tau$. After the payment of interest due in period t, a bond issued in period $t-\theta$, coming to maturity in period $t+\tau$, and bearing interest at rate $i_{t-\theta}^B$, has a capitalized value in $t+\tau$ of $(1+i_{t-\theta}^B)^\tau$. In equilibrium, a portfolio manager must be indifferent between these two securities. Their relative price in period t is therefore

$$\frac{\left(1+i_{t-\theta}^B\right)^\tau}{\left(1+i_t^B\right)^\tau}$$

Let us now compare two bonds issued in period t at face value, and bearing interest at the current rate i_t^B, one coming to maturity in t+1, and the other in $t+\tau$. Their relative price depends on expectations relative to interest rates which will prevail in t+1 and in every period until $t+\tau-1$. Under myopic expectations, the relative price of these two bonds will be 1, because the capitalized value is the same in both cases : the capitalized value of the bond coming to maturity in t+1 will be reinvested from period to period at the expected constant rate of i_t^B , until $t+\tau$.

So, in general, under myopic expectations relative to interest rates, the price of all bonds issued in the current period is the same. But the relative price of outstanding bonds (other than those issued in the current period) depends on the rate of interest of each one of them (and, therefore, on the period of issue) and on the number of periods remaining to maturity. The price at time t of a bond issued in $t-\theta$, coming to maturity in period $t+\tau$, and bearing interest at rate $i_{t-\theta}^B$, is given by

$$PO_t\ (\theta, \theta + \tau) = \frac{\left(1 + i_{t-\theta}^B\right)^\tau}{\left(1 + i_t^B\right)^\tau} \qquad [001]$$

As expected, the price of bonds issued in period t ($\theta = 0$) is equal to 1. And the price of bonds maturing in period t ($\tau = 0$), which will be redeemed at face value, is also equal to 1.

In what follows, the price of bonds issued in t–1, and coming to maturity in t+1

$$PO_t\ (1,2) = \frac{\left(1 + i_{t-1}^B\right)}{\left(1 + i_t^B\right)} \qquad [002]$$

plays an important part. Indeed, we shall take advantage of the relation

$$\left(1 + i_{t-1}^B\right) = \left(1 + i_t^B\right) PO_t\ (1,2) \qquad [003]$$

which, by recursion, leads to

$$\left(1 + i_{t-\theta}^B\right) = \left(1 + i_{t-\theta+1}^B\right) PO_{t-\theta+1}(1,2) = \left(1 + i_{t-\theta+2}^B\right) PO_{t-\theta+2}(1,2) PO_{t-\theta+1}(1,2) = \cdots \qquad [004]$$

$$\left(1 + i_{t-\theta}^B\right) = \left(1 + i_t^B\right) \prod_{s=1}^{\theta} PO_{t-\theta+s}(1,2) \qquad [005]$$

To streamline notation, we define

$$P_t^B = PO_t\ (1,2) = \frac{\left(1 + i_{t-1}^B\right)}{\left(1 + i_t^B\right)} \qquad [006]$$

so that [005] is now written as

$$\left(1+i^{B}_{t-\theta}\right)=\left(1+i^{B}_{t}\right)\prod_{s=1}^{\theta}P^{B}_{t-\theta+s} \tag{007}$$

We also have the following equivalence

$$PO_t\left(\theta,\theta+\tau\right)=\frac{\left(1+i^{B}_{t-\theta}\right)^{\tau}}{\left(1+i^{B}_{t}\right)^{\tau}}=\left(\prod_{s=1}^{\theta}P^{B}_{t-\theta+s}\right)^{\tau} \tag{008}$$

1.2 Redemption of bonds at maturity

In the basic model, it is assumed that the maturity structure of issues in all periods is the same. But it is not assumed that the maturity structure is flat, i.e. that an equal fraction of the bonds issued in a given period come to maturity in each following period until there are none left. Rather, that fraction is defined separately for every term $m \leq M$, M being the longest maturity admitted in the model.

So it is assumed that a constant fraction f_m of the bonds issued in each period t is for a term of $m \leq M$ periods. Of course, consistency requires

$$\sum_{m=1}^{M}f_m = 1 \tag{009}$$

Let ΔB_t^{τ} be the stock of bonds issued in period t and coming to maturity in period $t+\tau$, and ΔB_t, the number of new bonds of any term issued in period t. It follows that

$$\Delta B_t^{m} = f_m \Delta B_t \text{, for } m \leq M \tag{010}$$

and, of course,

$$\sum_{m=1}^{M}\Delta B_t^{m} = \Delta B_t \tag{011}$$

With the term structure so specified, one can compute the amount to be paid in redemption in period t as

$$REMB_t = \sum_{\theta=1}^{M}\Delta B^{\theta}_{t-\theta} = \sum_{m=1}^{M}f_m \Delta B_{t-m} \tag{012}$$

It is admittedly somewhat restrictive to impose the assumption of a constant maximum term and maturity structure of issues. But that kind of restriction is unavoidable given the form of price equations [001] or [008] : for lack of a maximum term, the equations, developed below, of the amount of interest to be paid and of the value of outstanding bonds, would involve theoretically infinite summations. We have been unsuccessful in our efforts to reduce these infinite summations to closed analytic forms (such as can be done for a converging geometrical series). Because recursive dynamic models do not accommodate calculations that involve variables from an indefinitely long past, some form of simplifying hypothesis had to be made, and we think the assumption which is made here is a reasonable compromise.

Note that, although the redemption of bonds having reached maturity is a negative cash flow, it is not part of current government expenditure and does not affect the level of savings. Even if agents' asset-liability balance is not explicit in the model, it is obvious that the redemption of bonds at maturity is a simultaneous decrease of government assets and liabilities, together with a change of nature for one element of the households' portfolio.

1.3 Interest currently payable on bonds

According to the time structure laid out earlier, the total amount of interest which a debtor must pay in a given period is the sum of interests payable on the remainder of bonds issued in all previous periods and not yet redeemed (which includes bonds coming to maturity in the current period, but excludes those issued in the current period).

It is demonstrated in Appendix 1 that the amount of interest due on outstanding bonds is given by

$$\begin{aligned} INT_t &= \sum_{\theta=1}^{M} i_{t-\theta}^{B}\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} \\ &= \sum_{\theta=1}^{M}\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1} P_{t-s}^{B}\right)\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta}-\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta}\right] \end{aligned} \qquad [013]$$

where, by convention, $f_0 = 0$. Let us define outstanding *face-value* debt at the beginning of period t (before debt coming to maturity in period t is redeemed, and before new bonds are issued) as DN_t :

$$DN_t = \sum_{\theta=1}^{M}\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} \qquad [014]$$

And the amount of interest due on outstanding bonds can be written :

$$INT_t = \sum_{\theta=1}^{M} i_{t-\theta}^{B}\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} = \sum_{\theta=1}^{M}\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1} P_{t-s}^{B}\right)\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta}\right] - DN_t \qquad [015]$$

In accordance with national accounting principles, interests paid on public debt are a transfer, which is added here to other public transfers. Transfers made as interest payments on public debt are distributed among households in fixed shares (a simplifying assumption) :

$$TGINT_{h,t} = TG_{h,t} + \lambda_h^{PIN} INT_t \qquad [016]$$

where $TG_{h,t}$ is the exogenous amount of public transfers to household *h*, and INT_t is the amount of interests paid on the public debt.

Contrary to the redemption of bonds having come to maturity, the payment of interests is part of current expenditures and affects government savings.

1.4 Aggregate value of outstanding bonds

The aggregate value of outstanding bonds issued in past periods is not the same thing as face-value debt. The latter is the sum of the face values of bonds issued in the past and not yet redeemed. The aggregate value of outstanding bonds, on the other hand, is measured without reference to the face value of bonds issued in the past : rather, it is the current market value of all outstanding bonds, except those issued in the current period, after redemption of those which have come to maturity.

Once bonds coming to maturity have been redeemed, the market value of bonds issued in the past and still outstanding in period *t* is given by

$$B_t = \sum_{\theta=1}^{M-1} \sum_{\tau=1}^{M-\theta} PO_t\left(\theta, \theta+\tau\right) \Delta B_{t-\theta}^{\theta+\tau} \qquad [017]$$

where the price at time *t* of a bond issued in $t-\theta$, coming to maturity in $t+\tau$, and bearing interest at rate $i_{t-\theta}^{B}$ has already been shown to be

$$PO_t\left(\theta, \theta+\tau\right) = \frac{\left(1+i_{t-\theta}^{B}\right)^{\tau}}{\left(1+i_t^B\right)^{\tau}} = \left(\prod_{s=1}^{\theta} P_{t-\theta+s}^{B}\right)^{\tau} \qquad [008]$$

and where

$$\Delta B_t^m = f_m \Delta B_t \text{, for } m \le M \qquad [010]$$

Therefore,

$$B_t = \sum_{\theta=1}^{M-1} \sum_{\tau=1}^{M-\theta} \left(\prod_{s=1}^{\theta} P^B_{t-\theta+s} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta} \qquad [018]$$

Because of the τ appearing in the exponent, it seems impossible to express B_t as a function of current prices and B_{t-1}. That is the reason why we could not avoid making an assumption about the maximum term, since, otherwise, the model's memory would be overwhelmed.

2. HOUSEHOLD PORTFOLIO ALLOCATION AND ASSET DEMAND

2.1 Household wealth

Households own a portfolio consisting of bonds and shares[4]. In every period, they reallocate all of their portfolio. What we call household wealth here is the value of the portfolio they have to allocate at the beginning of each period. The *ex ante* value of the portfolio is equal to the value of shares and outstanding bonds (excluding those coming to maturity in the current period) which households own at the beginning of the period, plus the amount received in redemption of bonds coming to maturity in the current period, and current savings :

$$W_t = A_t + B_t + REMB_t + \sum_h SM_{h,t} \qquad [019]$$

where

A_t is the market value of shares held by households at the beginning of period t

$SM_{h,t}$ is savings of household category h in period t

and $REMB_t$ and B_t are given by [012] and [018] respectively.

Note that A_t does not include currently issued shares ΔA_t. Nor is it the market value of *all* shares outstanding at the beginning of period t : rather, it is the market value only of those shares owned by households (it would therefore be *erroneous* to beleive that the stock of outstanding shares evolves according to the simple rule $A_t = A_{t-1} + \Delta A_t$).

A_t is a share of the market value of all capital inherited from the past WK_t :

4 Note that in this simplified model, households do not own physical capital (such as housing). In effect, housing assets are implicitly owned in the form of shares of the « Owner-occupied dwellings » industry which is part of the services sector. This implies that housing is treated as a perfect substitute to shares, an assumption justified only by the « didactical » nature of EXTER-Debt.

$$A_t = \lambda_{H,t}^{K} WK_t \qquad [020]$$

We shall see later on how $\lambda_{H,t}^{K}$ and A_t are determined (equations [076] and [080]).

2.2 Household asset demand

Portfolio allocation follows the Decaluwé-Souissi model (1994; Souissi, 1994; Souissi and Decaluwé, 1997). Let A_t^* and B_t^* be the values of share and bond asset holdings respectively, after portfolio reallocation; also let i_t^A be the rate of return on shares. The portfolio manager maximizes the capitalized value of his/her holdings at the beginning of the following period (when interests and dividends are paid) :

$$\underset{A_t, B_t}{MAX}\ VC = \left(1 + i_t^A\right) A_t^* + \left(1 + i_t^B\right) B_t^* \qquad [021]$$

subject to the total value of his/her portfolio

$$A_t^* + B_t^* = W_t \qquad [022]$$

where W_t is given by [019], and subject to CET asset aggregation function

$$W_t = A_W \left[\delta_A \left(A_t^*\right)^{\beta} + \delta_B \left(B_t^*\right)^{\beta} \right]^{\frac{1}{\beta}} \qquad [023]$$

with elasticity of transformation

$$\tau = \frac{1}{1-\beta} \ (\beta > 1) \qquad [024]$$

A few words about [023]. First, note that, if it were not for that constraint, all of the portfolio would be allocated to the asset with the highest rate of return in utility function [021]. Aggregation function [023], concave to the origin, imposes diversification (except in the particular case of corner solutions).

But accounting consistency requires that wealth constraint [022] be simultaneously satisfied. As demonstrated in Appendix 2, that implies

$$A_W = \left\{ \delta_A^{\ \tau} \left[\left(1 + i_t^A\right)\right]^{-\tau} + \delta_B^{\ \tau} \left(1 + i_t^B\right)^{-\tau} \right\} \left\{ \delta_A^{\ \tau} \left(1 + i_t^A\right)^{1-\tau} + \delta_B^{\ \tau} \left(1 + i_t^B\right)^{1-\tau} \right\}^{\frac{\tau}{1-\tau}} \qquad [025]$$

Whence, demand functions

$$A_t^* = W_t \frac{\delta_A{}^\tau \left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^\tau \left(1+i_t^A\right)^{-\tau} + \delta_B{}^\tau \left(1+i_t^B\right)^{-\tau}\right\}} \qquad [026]$$

and

$$B_t^* = W_t \frac{\delta_B{}^\tau \left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^\tau \left(1+i_t^A\right)^{-\tau} + \delta_B{}^\tau \left(1+i_t^B\right)^{-\tau}\right\}} \qquad [027]$$

The model can do without scale variable A_W.

Our formulation differs from the Decaluwé-Souissi model in two respects. First, A_t^* and B_t^* are the *values* of shares and bonds respectively, not their quantities . This implies that their prices are 1 by definition. Price variations appear as changes in rates of return, and are reflected in the value computation formulae applied to past issues (see 1.4 above, and 2.3 below). The second difference follows from the first : diversification constraint [023] is formulated in terms of asset values, rather than quantities.

3. ASSET MARKET EQUILIBRIUM

3.1 The supply of bonds

In each period, the supply of bonds consists of two components : the value of newly issued bonds, and that of bonds which were outstanding in the preceding period and do not come to maturity in the current period. New bonds are issued at a fixed price of 1; it is the current rate of interest which adjusts so as to equilibrate supply and demand[5]. As we have shown above, the price of bonds issued in previous periods is adjusted according to the current rate of interest, so that bond-holders are indifferent between new bonds and those which were already outstanding. The aggregate *value* of still outstanding, previously issued bonds is computed with the adjusted prices in equation [018][6].

The supply of new bonds is determined by the government's financing needs. Net government financing needs are the difference between, on one hand, the total value of public investments

5 This is the exact opposite of Treasury bills, whose yield depends on the acquisition-price discount relative to its face value.

6 One can imagine that in every period, old bonds still outstanding are revalued according to [018], and then exchanged for an equal value of replacement bonds worth 1.

and redemption of debt coming to maturity in the current period, and, on the other hand, current government savings[7] :

$$\Delta B_t = PK_t IG_t + REMB_t - SG_t \qquad [028]$$

where

IG_t is public investment in real terms in period t

PK_t is the price of investment goods in period t

SG_t is government savings in period t

It is not excluded *a priori* that the supply of new bonds be negative.

We make the simplifying assumption that public investments are a constant share of total investment :

$$PK_t IG_t = \pi_{IP} IT_t \qquad [029]$$

where IT_t is total investment spending in period t, so that

$$\Delta B_t = \pi_{IP} IT_t + REMB_t - SG_t \qquad [030]$$

This simplifying assumption could easily be replaced by a more realistic specification, such as a public investment program (fixed exogenous values).

3.2 Stock market

In each period, the total value of shares offered consists of the value of newly issued shares, and shares issued in the past.

The value of new shares issued is equal to business financing requirements, which are the difference between private investment expenditure and business savings. Thus, we make the simplifying assumption that all private investment that is not financed out of business savings is financed by issuing new shares :

$$\Delta A_t = PK_t IE_t - SE_t \qquad [031]$$

where

IE_t is private investment in real terms in period t

PK_t is the price of investment goods in period t

SE_t is business savings in period t

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7 Of course, government savings are calculated taking into account interest payments on the public debt.

Given [029],

$$PK_t IE_t = \left(1 - \pi_{IP}\right) IT_t \qquad [032]$$

and

$$\Delta A_t = \left(1 - \pi_{IP}\right) IT_t - SE_t \qquad [033]$$

It is not excluded *a priori* that the supply of new shares be negative.

3.3 Asset supply and demand equilibrium

Given that EXTER-Debt is developed essentially for illustrative purposes, and that we want to keep it as simple as possible, even at some cost in terms of realism, we make the following additional assumption. Foreign savings, equal to current account deficit CAB_t, are entirely dedicated to the purchase of shares; only the remainder, $\Delta A_t - CAB_t$, is offered to households. In a CGE based on real data, the RoW would own a portfolio, just like households. The RoW's portfolio would consist minimally of domestic shares and bonds, and of at least one foreign asset (labeled in a foreign currency). Model closure could then be modified. In EXTER-Debt, the current account deficit is fixed exogenously. But if the RoW owned a portfolio, the current account balance would be linked to the value of domestic assets which the RoW agrees to include in its portfolio, depending on the rate of return on the foreign asset : that foreign rate of return could then be fixed exogenously to close the model.

Given this additional simplifying assumption, household demand for shares must absorb the market value of shares already owned at the beginning of the period, A_t, plus newly issued shares, ΔA_t, minus foreign acquisition of shares, assumed to be equal to current account balance CAB_t. So we have

$$A_t^* = A_t + \Delta A_t - CAB_t \qquad [034]$$

Whence, given [026],

$$A_t + \Delta A_t - CAB_t = W_t \frac{\delta_A{}^{\tau}\left(1 + i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1 + i_t^B\right)^{-\tau}\right\}} \qquad [035]$$

Household bond demand must absorb all supply of bonds, which consists of bonds issued in the current period (at face value), plus the value of bonds issued in the past and not coming to maturity in the current period. Hence,

$$B_t^* = B_t + \Delta B_t \qquad [036]$$

and, given [027],

$$B_t + \Delta B_t = W_t \frac{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [037]$$

The interest rate adjusts in order to equilibrate the bond market.

3.4 Savings-investment equilibrium

Savings-investment equilibrium is characterized in our model by

$$IT_t = \sum_h SM_{h,t} + SE_t + SG_t + CAB_t \qquad [038]$$

That condition is automatically satisfied by asset market equilibrium. Indeed, given household asset demands [035] and [037],

$$B_t + \Delta B_t + A_t + \Delta A_t - CAB_t = W_t \qquad [039]$$

Substituting from household wealth constraint [019] results in

$$B_t + \Delta B_t + A_t + \Delta A_t - CAB_t = W_t = A_t + B_t + REMB_t + \sum_h SM_{h,t} \qquad [040]$$

$$\Delta B_t + \Delta A_t - REMB_t = CAB_t + \sum_h SM_{h,t} \qquad [041]$$

Substituting the supply of new bonds [030] and shares [033], yields

$$\left[\pi_{IP} IT_t + REMB_t - SG_t\right] + \left[\left(1-\pi_{IP}\right) IT_t - SE_t\right] - REMB_t = CAB_t + \sum_h SM_{h,t} \qquad [042]$$

which is equivalent to [038].

4. INVESTMENT DEMAND AND EQUILIBRIUM MECHANISM

4.1 Investment demand

Let

$r_{i,t}$ be the rental rate of industry *i*'s capital in period *t*

tye be the (marginal) rate of taxation applied to capital income before depreciation, so the after-taxes rental rate is (1–*tye*) $r_{i,t}$

δ_i be the rate of depreciation of industry *i*'s capital

ϕ_t be the market discount rate applied in period *t*

Then the present value of the income stream generated by one unit of capital, beginning in $t + 1$ at $r_{i,t}$, and declining thereafter at a rate of δ_i per period, is equal to

$$\frac{1}{1-\delta_i}\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}(1-tye)r_{i,t}=\frac{(1-tye)r_{i,t}}{\phi_t+\delta_i} \qquad [043]$$

Now let PK_t be the replacement price of capital in period *t*, and

$$U_{i,t}=PK_t\left(\phi_t+\delta_i\right) \qquad [044]$$

is industry *i*'s user cost of capital in period *t*.

Then

$$\frac{(1-tye)r_{i,t}}{PK_t\left(\phi_t+\delta_i\right)}=\frac{(1-tye)r_{i,t}}{U_{i,t}} \qquad [045]$$

is the ratio of the market value to the replacement cost of a unit of capital, and it can be interpreted as Tobin's « *q* ». Investment demand is specified following Jung and Thorbecke (2001)[8] as a constant elasticity increasing function of Tobin's « *q* » :

$$\frac{Id_{i,t}}{KD_{i,t}}=\gamma 1_i\left(\frac{(1-tye)r_{i,t}}{U_{i,t}}\right)^{el\_ind_i} \qquad [046]$$

It is acknowledged that this specification is at variance with Tobin's theory. Indeed, according to the « *q* » theory, equilibrium investment is such that *q* equals 1.

Let the gross rate of return on capital in industry *i*, before depreciation, but net of taxes on capital income[9], be

$$\rho_{i,t}=\frac{(1-tye)r_{i,t}}{PK_t} \qquad [047]$$

and assume the expected return to be constant, equal to $\rho_{i,t}$ (myopic expectations). Also define

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8 Our specification differs from Jung-Thorbecke in that depreciation is taken into account in [044].

9 This qualification is of no practical relevance in EXTER-Debt, where there are no such taxes.

$$\zeta_{i,t} = \frac{U_{i,t}}{PK_t} = \phi_t + \delta_i = \frac{1}{\frac{1}{1-\delta_i}\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}} = \frac{1-\delta_i}{\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}} \quad [048]$$

The present value of a stream of income beginning in $t + 1$ at $\zeta_{i,t}$, and declining thereafter at a rate of δ_i per period, discounted at rate ϕ_t, is equal to 1. So $\zeta_{i,t}$ is the rate of return required for the present value of the income stream generated by an investment in industry i to be equal to its cost when the market discount rate ϕ_t is applied.

From [047] and [048], investment demand [046] can be rewritten as

$$\frac{Id_{i,t}}{KD_{i,t}} = \gamma 1_i \left(\frac{(1-tye)r_{i,t}}{U_{i,t}}\right)^{el\_ind_i} = \gamma 1_i \left(\frac{\rho_{i,t}}{\zeta_{i,t}}\right)^{el\_ind_i} \quad [049]$$

The present value of the stream of income expected from new investment in industry i is larger or smaller than its cost (Tobin's « q » is above or below 1), according to whether $\rho_{i,t}$ is more or less than $\zeta_{i,t}$.

Equilibrium between investment demand and total investment spending imposes the following constraint :

$$IT_t = PK_t \sum_i Id_{i,t} \quad [050]$$

Note that the implicit assumption in the preceding equation is that public investment, which is a constant share of total investment according to [029], is distributed among industries together with private investment. This is another one of our simplifying assumptions, and it could easily be relaxed. We shall further assume that government receives shares for its contribution, like any other agent.

4.2 A closer look at the user cost of capital

4.2.1 NEW SHARE ISSUES

Buyers of new shares in industry i will demand the market rate of return $\zeta_{i,t}$. Specifically, new shareholders will demand a number of shares that will entitle them to a fraction of the income generated by the industry that will be sufficient for them to realize the yield they expect. So, to raise new capital to finance investments of $PK_t\ Id_{i,t}$, the number of new shares issued will have to be

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} (1 - tye) r_{i,t}\ KD_{i,t+1} = \zeta_{i,t}\ PK_t\ Id_{i,t} \qquad [051]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} (1 - tye) r_{i,t}\ \left[(1 - \delta_i) KD_{i,t} + Id_{i,t} \right] = \zeta_{i,t}\ PK_t\ Id_{i,t} \qquad [052]$$

where $N_{i,t}$ is the number of shares outstanding at the beginning of the period, and $\Delta N_{i,t}$ the number of new shares issued.

Equation [052] reflects the view that, in accordance with the time structure adopted, income generated in the current period goes to those who were shareholders at the beginning of the period. Thus, when one purchases ownership of one unit of capital in industry i in period t, he/she will begin receiving income from period $t + 1$ onwards. So the left-hand side of [052] is the amount of income that will be paid to holders of the new shares in period t + 1, according to their share of ownership, assuming the rental price of capital $r_{i,t}$ is the same as in the current period (myopic expectations). The right-hand side is the income they must receive for the present value of expected incomes to be equal to the value of the investment under myopic expectations. In other words, the number of shares issued must be such that the fraction of expected income attributed to the holders of new shares will produce the income they demand.

Substitute

$$\rho_{i,t} = \frac{(1 - tye) r_{i,t}}{PK_t} \qquad [047]$$

and [052] becomes

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} \rho_{i,t} PK_t\ \left[(1 - \delta_i) KD_{i,t} + Id_{i,t} \right] = \zeta_{i,t}\ PK_t\ Id_{i,t} \qquad [053]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} \rho_{i,t}\ \left[(1 - \delta_i) KD_{i,t} + Id_{i,t} \right] = \zeta_{i,t}\ Id_{i,t} \qquad [054]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} = \frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1 - \delta_i) KD_{i,t} + Id_{i,t}} \qquad [055]$$

So the equity owned by new investors is more or less than their contribution to the industry's capital in t + 1 according to whether $\zeta_{i,t}$ is greater or smaller than $\rho_{i,t}$: if $\zeta_{i,t}$ is greater than $\rho_{i,t}$, new investment is financed by equity dilution; in the opposite case, there is equity enhancement.

So, for every dollar invested, the new shareholder receives an expected income stream of $\zeta_{i,t}$ in period t + 1, declining thereafter at the rate of δ_i. But the relevant rate for the portfolio manager is the rate of return *net of depreciation*, i_t^A , which is defined implicitly by

$$\sum_{\theta=1}^{\infty} \frac{1}{(1+\phi_t)^{\theta}} i_t^A = \sum_{\theta=1}^{\infty} \left(\frac{1-\delta_i}{1+\phi_t} \right)^{\theta} \zeta_{i,t} = 1 \qquad [056]$$

The present value of a *constant* stream of income of i_t^A is equal to that of an expected income stream of $\zeta_{i,t}$ in period t + 1, declining thereafter at the rate of δ_i. Since

$$\sum_{\theta=1}^{\infty} \frac{1}{(1+\phi_t)^{\theta}} = \frac{1}{\phi_t} \qquad [057]$$

we substitute in [056] :

$$\sum_{\theta=1}^{\infty} \frac{1}{(1+\phi_t)^{\theta}} i_t^A = \frac{1}{\phi_t} i_t^A = 1 \qquad [058]$$

Therefore

$$i_t^A = \phi_t. \qquad [059]$$

The market rate of return which guides the portfolio manager in his/her choices, i_t^A , is none other than the market discount rate ϕ_t.

It also follows from [044] and [048] that

$$U_{i,t} = PK_t \zeta_{i,t} = PK_t (\phi_t + \delta_i) = PK_t (i_t^A + \delta_i) \qquad [060]$$

The key role played by market rate of return i_t^A in the savings-investment equilibrating mechanism is now clear. Through user cost of capital $U_{i,t}$, any rise in i_t^A dampens investment demand [046], which reduces the amount of new share issues ΔA_t (equations [031], [032], and

[050]). A rise in i_t^A also increases the fraction of the household portfolio dedicated to shares (demand equation [026]), while reducing the market value of shares issued in the past already held in the portfolio A_t (equations [020] and [076]), and total household wealth [019]. The net effect on new share demand by households is nonetheless positive, as shown in Appendix 4).

4.2.2 NEW AND OLD SHARE PRICES AND THE STOCK MARKET VALUATION OF CAPITAL

Since a number $\Delta N_{i,t}$ of new shares have been issued for a total amount of $PK_t\, Id_{i,t}$, their unit price is

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} \tag{061}$$

As for the holders of shares outstanding at the beginning of the period, it follows from [052] that what they must give up to the new shareholders is $\zeta_{i,t}\, PK_t\, Id_{i,t} = U_{i,t}\, Id_{i,}$, which is the user cost of the new capital, as it should. So, assuming myopic expectations, they expect to receive an income in period t + 1 of

$$\rho_{i,t} PK_t \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] - \zeta_{i,t}\ PK_t\ Id_{i,t} = PK_t\left\{\rho_{i,t}\left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] - \zeta_{i,t}\ Id_{i,t}\right\} \tag{062}$$

$$\rho_{i,t} PK_t \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] - \zeta_{i,t}\ PK_t\ Id_{i,t} = PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t} + \left(\rho_{i,t} - \zeta_{i,t}\right)Id_{i,t}\right] \tag{063}$$

In the following periods, that stream of income declines with depreciation. Since new and old shareholders are the same, their discount rates are the same : i_t^A = $\phi_{i,t}$. So the present value of the stream of income that old shareholders expect to receive is equal to :

$$\frac{1}{(1-\delta_i)}\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+i_t^A}\right)^{\theta} PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t} + \left(\rho_{i,t} - \zeta_{i,t}\right)Id_{i,t}\right]$$

$$= \frac{1}{i_t^A + \delta_i} PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t} + \left(\rho_{i,t} - \zeta_{i,t}\right)Id_{i,t}\right] \tag{064}$$

$$= \frac{1}{\zeta_{i,t}} PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t} + \left(\rho_{i,t} - \zeta_{i,t}\right)Id_{i,t}\right] \tag{065}$$

$$= PK_t \left[\frac{\rho_{i,t}}{\zeta_{i,t}} (1-\delta_i) KD_{i,t} + \left(\frac{\rho_{i,t}}{\zeta_{i,t}} - 1 \right) Id_{i,t} \right] \quad [066]$$

$$= PK_t \left[\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right] \quad [067]$$

The unit price of old shares is their value, divided by their number :

$$\frac{PK_t}{N_t} \left[\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right]$$

It is demonstrated in the following box that the price of old shares is identical to the price of new ones. So share pricing is perfectly compatible with the fact that new and old shareholders are the same agent.

Demonstration of the equality of prices between old and new shares :

From [055], the number of new shares is

$$\Delta N_{i,t} = \frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1-\delta_i) KD_{i,t} + Id_{i,t}} \left(N_{i,t} + \Delta N_{i,t} \right) \quad [068]$$

$$\left(1 - \frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1-\delta_i) KD_{i,t} + Id_{i,t}} \right) \Delta N_{i,t} = \frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1-\delta_i) KD_{i,t} + Id_{i,t}} N_{i,t} \quad [069]$$

$$\Delta N_{i,t} = \frac{\left(\frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1-\delta_i) KD_{i,t} + Id_{i,t}} \right)}{\left(1 - \frac{\zeta_{i,t}}{\rho_{i,t}} \frac{Id_{i,t}}{(1-\delta_i) KD_{i,t} + Id_{i,t}} \right)} N_{i,t} \quad [070]$$

$$\Delta N_{i,t} = \frac{1}{\left(\frac{\rho_{i,t}}{\zeta_{i,t}} \frac{(1-\delta_i) KD_{i,t} + Id_{i,t}}{Id_{i,t}} - 1 \right)} N_{i,t} \quad [071]$$

It follows that the price of new shares can be written as

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t\ Id_{i,t}}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} \frac{(1-\delta_i)KD_{i,t} + Id_{i,t}}{Id_{i,t}} - 1 \right) \qquad [072]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] - Id_{i,t} \right) \qquad [073]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right) \qquad [074]$$

which is precisely the price of old shares.

Q.E.D.

Since the price of old and new shares is the same, the total stock market value of industry *i* in period *t* is simply

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} \Delta N_{i,t} + \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right) N_{i,t} = PK_t \frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} \qquad [075]$$

Note that the current stock market value of an industry is different from the market value of its currently productive capital, since the stock market value is forward looking in that it includes the value of current investments which come on line only in the following period, and takes into account end-of-period depreciation. But the stock market value is forward looking in a simplistic way, since it is based on myopic expectations relative to replacement price of capital PK_{t+1}, and return rates $\rho_{i,t+1}$ and $\zeta_{i,t+1}$.

The second term on the left-hand side of [075] is the current stock market value of shares issued in the past :

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\zeta_{i,t}} (1-\delta_i)KD_{i,t} + \left(\frac{\rho_{i,t}}{\zeta_{i,t}} - 1 \right) Id_{i,t} \right] \qquad [076]$$

4.2.3 OWNERSHIP AND DIVIDEND DISTRIBUTION

Contrary to stock market value, dividend distribution is, so to speak, backward looking : capital income is generated by currently productive capital, which consists of depreciated capital that was already productive, and investments made in the preceding period.

So the fraction of current capital income generated in industry *i* that belongs to those who invested in the previous period is

$$\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}$$

The fraction of ownership attributed to new shares issued in period *t*–1, according to [055], corresponds to the whole amount of investment. But part of that was financed by business savings, retained earnings of the previous period, which were not distributed as dividends. So the corresponding fraction of ownership must be distributed proportionately to dividend entitlements in the previous period. Therefore, given that the fraction of investment finances out of retained earnings in period *t*–1 is $\frac{SE_{t-1}}{IT_{t-1}}$, the ownership fraction of agent *j* in the capital of industry *i* during period *t* is therefore

$$\lambda^{K}_{j,t-1}\left(\frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}+\frac{SE_{t-1}}{IT_{t-1}}\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}\right)+\lambda^{I}_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} \quad [077]$$

where it is assumed that, in period *t*–1, the ownership fraction of a given agent was the same in all industries (as if the shares of all industries were pooled in a single mutual fund), and where $\lambda^{I}_{j,t-1}$ is the fraction acquired by agent *j* in the new shares issued in period *t*–1 that correspond to the part of investment not financed out of retained earnings ($\lambda^{I}_{j,t-1}$ is also assumed to be equal across industries).

To be consistent with our simplifying assumption that public investment funding is pooled with private funding (equations [029] and [050]), the government is also a shareholder, whose purchases of new shares are equal to public investment expenditures :

$$\lambda^{I}_{G,t-1}=\frac{\pi_{IP}IT_{t-1}}{IT_{t-1}-SE_{t-1}} \quad [078]$$

And, consistent with [034], the RoW's share in investments is

$$\lambda^{I}_{RoW,t-1}=\frac{CAB_{t-1}}{IT_{t-1}-SE_{t-1}} \quad [079]$$

Finally, the households' share is

$$\lambda^{I}_{H,t}=1-\lambda^{I}_{RoW,t}-\lambda^{I}_{G,t} \quad [080]$$

The ownership fraction of agent *j* in the capital of all industries (in the all-industries mutual fund), $\lambda^K_{j,t}$, is the weighted sum of industry fractions [077]. It is demonstrated in Appendix 3 that

$$\lambda^K_{j,t} = \frac{\lambda^K_{j,t-1}\left(WK_{i,t-1} + SE_{t-1}\right) + \lambda^I_{j,t-1}\left(IT_{t-1} - SE_{t-1}\right)}{WK_{i,t-1} + IT_{t-1}} \qquad [081]$$

5. SIMULATION EXPERIMENTS WITH EXTER-DEBT

5.1 Calibration

In the version presented here, EXTER-Debt calibrated with the objective that, in the absence of shocks, it follow a regular path, that is, a path of balanced growth, at a rate equal to exogenous population growth *n*.

A necessary condition to achieve that purpose, is that exogenous flows all grow at rate *n* :

- labor supply

  $$LS_{t+1} = (1+n)LS_t \qquad [082]$$

- public transfers to households (except interests paid on the public debt, which are endogenous)

  $$TG_{h,t+1} = (1+n)TG_{h,t} \qquad [083]$$

- government spending

  $$G_{t+1} = (1+n)G_t \qquad [084]$$

- the current account balance

  $$CAB_{t+1} = (1+n)CAB_t \qquad [085]$$

A second requirement is that past bond issues must have grown at the demographic growth rate *n*, so that the amount to be redeemed may grow at the same rate in the future. Government savings depend on the amount of interests paid on outstanding bonds, which in turn depends on the history of bond issues. For interest payments on the public debt to grow at rate *n*, with amounts issued also growing at rate *n*, past period interest rates must have been equal to the current base period rate. Finally, given the amount of government savings, and a current issue of new bonds equal to *n*+1 times that of the previous period, public investment is set accordingly. It must also be taken into account that investment expenditures depend, among others, on household savings, and therefore on interest payments on the public debt.

A further requirement of a regular path is that the capital stock and investment demand functions be calibrated so that, *coeteris paribus*, the capital stock of every industry grows at rate n. Moreover, the initial capital ownership shares of agents were set equal to their respective contributions to investment in the base period so that ownership shares remain constant. The initial rate of return on shares was set equal to the weighted average rate of return to capital, net of taxes and depreciation.

It was not easy to engineer a SAM with all the required characteristics. And, as we shall see later on, although these characteristics are necessary conditions for the economy to follow a regular path, they are not quite sufficient.

5.2 Simulations

5.2.1 QUASI-REGULAR PATH

We have run our model without shock for 30 periods. We observe that the period-to-period change in flow variables is indeed very close to the 1% demographic rate of growth stipulated. But not exactly. And most price variables change very little. But they do. So our results reveal that the path followed by the economy is not quite regular.

Why? Among the price variables that do change significantly is the rate of interest. After 30 periods, the rate of return on shares, `irac`, is practically unchanged (+0.33% relative to its base period value), while interest rate `ir` has increased by more than 55%. That is because our calibration procedure has not taken into account the initial proportion of shares and bonds in the households' portfolio. Now, the ratio of the value of new to old bonds in the portfolio is far larger than that of new to old shares. So the proportion of each type of asset in the portfolio changes progressively in favor of bonds, which exerts pressure on the interest rate : households are persuaded to increase the proportion of bonds in their portfolio only if they are offered a higher rate of interest.

It is not so easy as it would seem at first sight to imagine numbers that would correct that situation. First, for new issues to grow at rate n, any increase in the stock of previously issued bonds implies a proportional increase of the issue of new bonds in the base period : consequently, this is no way to move the rates of increase of the two portfolio components closer to one another. As for modifying the value of A_t, shares issued in previous periods in the household portfolio, it is precluded by the tight relationship between A_t and WK_t (equation [020]), and by the fact that the latter is entirely determined by previously calibrated variables (see A5.3 in Appendix 5). Finally, modifying the amount of new shares issued is equivalent to constructing a whole new SAM.

In view of these difficulties, we shall be content with a quasi-regular path.

5.2.2 Zero-growth scenario

Before proceeding with counter-factual simulations departing from the quasi-regular path, the model was further tested to see if it could reproduce a stationary state, i.e. a situation in which the base-period values are repeated from period to period, indefinitely. Basically, the necessary conditions are the same as for the quasi-regular path, except that the growth rate *n* is set to zero.

This exercise required constructing a new SAM, because of changes in the history of past bond issues. For a stationary state to prevail, past issues must have been of equal amounts every period. And that implies that new bonds are currently issued for an amount exactly equal to current redemption of bonds coming to maturity; this differs from the quasi-regular SAM, where the amount of newly issued bonds is n+1 times the redemption of mature bonds. It follows that public investment is equal to government savings. So adjustments cascade throughout the SAM. To facilitate the construction of the steady-state SAM, dividend distribution parameters were disconnected from ownership fractions, and simply set equal to initial values.

Apart from that, and starting from a different SAM, the zero-growth version of EXTER-Debt is identical to the other one[10]. And, as expected, the base-period values are repeated from period to period, indefinitely.

5.2.3 Doubling the share of public investments

The BAU scenario is the quasi-regular path described above. The shock is applied in the second period. The amount of new bonds issued immediately increases by more than 80%, while there are consequently less new shares issued. In order for the households to absorb the abundance of new bonds, the rate of interest jumps from 1% to nearly 2.5%. As a result, interest payments begin to increase in the following, third, period. Government savings show a modest increase in the third period, a paradox which disolves when it is realized that the increase in public investment augments the government ownership fraction, and therefore the amount of dividends it receives. But that effect is quickly overwhelmed, beginning in the fourth period, by

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[10] Full details are available on request.

ever increasing interest payments on the public debt, both because larger bond issues inflate the public debt, and because of a higher interest rate. By period 13, the government is running a deficit, the interest rate has risen to almost 12% (whereas the rate of return on shares hasn't moved very much), and the current new bond issue is more than sixfold its first-period amount. The model collapses in period 21.

One may wonder why this happens. After all, in our minimalist model, all the government does is borrow to channel the money into investment : so why should it matter that the government's share of total investment increase? Part of the explanation is that interests paid on the public debt reduce government savings dollar for dollar, while interest transfer payments to households are saved in part only. Also, the ballooning of interest payments is accelerated by the strong portfolio reallocation effect generated by our artificial data. The fraction of the household portfolio initially occupied by bonds is just under 4%; it increases to more than 65% in period 21, the final one before melt-down. Such a radical change in proportions is possible only with spectacular changes in the relative rates of return.

5.2.4 NO PUBLIC INVESTMENTS

The opposite scenario, zero public investments, collapses even more quickly, in period 5. With positive government savings and no public investment, new issues of bonds fall rapidly, and become negative in period 4. By then, the household portfolio consists almost exclusively of shares, and in the following period, there is no portfolio allocation consistent with a negative amount of outstanding bonds.

5.2.5 FIVE PERCENT TARIFF REDUCTION

This more conventional simulation produces the expected results. Government revenue falls slightly, rather than increasing at the 1% demographic growth rate. Government savings fall too, and new bond issued jump by 10%. Investment is reduced, since the drop in government savings is not compensated by other agents. In the end, after 30 periods, the stock of capital is lower than in the BAU quasi-regular path scenario : 5.9% less in agriculture, 7.6% less in industry, and 3.9% less in tradable services.

SUMMARY AND CONCLUSIONS

In this paper, we have presented a minimalist version of a model of bond indebtedness. The proposed specification takes into account the following characteristics of bonds :

- they are issued at a given date;
- they have a given nominal, or face value;
- they bear interest at a given rate relative to their face value;
- they have a maturity date, at which they are reimbursed by the issuer to the holder.

Bonds compete with another type of asset, so that the yield demanded by the buyers of new bonds increases as the stock of bonds grows relative to the stock of outstanding shares.

Restrictions to the maturity structure of bonds make it possible to achieve a reasonable compromise between a realistic representation of the evolution of bond debt, and the weight of past variable values which the model must keep in memory.

In the proposed model, the government borrows by issuing bonds, redeems bonds when they reach maturity, and pays interest on outstanding debt. The prices of bonds issued at different times with different maturity terms are consistent with an arbitrage equilibrium. The supply of new bonds and shares is determined by the financing needs of government and of business. Asset demand reflects the rational behavior of households managing their portfolio, according to the Decaluwé-Souissi model.

The model specification outlined is illustrated in the EXTER-Debt model, which was developed to demonstrate the feasibility of the modeling principle presented in this paper. In EXTER-Debt, we make several assumptions to make the model as simple as possible, even at some cost in terms of realism, so as not to distract from the illustration of the proposed modeling approach. The main simplifying assumptions are :

A1 No risk aversion and myopic expectations.

A2 The term structure of bond issues is the same in all periods : it is assumed that a constant fraction f_m of securities issued in each period t has a maturity term of $m \leq M$, where M is the maximum possible maturity term.

A3 Transfers made as interest payments on public debt are distributed among households in fixed shares

A4 Public investments are a constant share of total investment.

A5 All investment that is not financed out of business savings, including public investment, is financed by issuing new shares.

A6 So-called « public investments » are actually public financing of private investments, insofar as public investments in any given period are simply added to private investments, and are distributed by destination industry in the same proportions as private investments. The government is assumed to receive shares for its contribution.

A7 The Rest-of-the-World (RoW) does not own a portfolio; all of foreign savings (current account deficit) is dedicated to purchasing shares.

All of the above assumptions, except A1 and A2, could easily be dispensed with, at the cost of some complexity. A2 is inevitable, but, given sufficient computing power, can be made less restrictive by extending the maximum term *M*.

Simulation experiments were performed with EXTER-Debt, based on artificial data. The artificial data was designed so that, in a BAU scenario, the model may generate a regular path, that is, a path of balanced growth, at a rate equal to exogenous population growth *n*. We were successful in developing an artificial data base that produces a quasi-regular path, which demonstrates one aspect of the model's consistency.

In spite of the artificial character of the underlying data, and in spite of several simplifying assumptions justified only by the illustrative nature of this version of the model, the simulation results are satisfying in that its behavior under various scenarios is in line with what theory would lead us to expect. True, this illustrative version of the model is far too simple for it to be able to absorb any shock. But even when the model collapses or explodes, the economic logic that leads to that result is clear.

Governments face difficult choices regarding the basic public finance identity :

Expenditures – Taxes = Deficit = Debt increase

The impact of expenditures was first analysed with Keynesian macro and input-output models. Then CGE models provided a tool to better understand the distortional effects of taxes. In this paper, we have proposed an approach to extend the CGE methodology to simulate the dynamics of debt. We hope our approach will be implemented and tested in a not-too-distant future, and will eventually contribute, modestly, to better public policy.

REFERENCES

Collange, Gérard (1993) *Un modèle de l'économie ivoirienne. Vol. 1: Synthèse et présentation économique*, CERDI.

Decaluwé, Bernard and Yvan Decreux (2004) « Rapport de mission d'appui au Ministère des Finances, Rabat, Maroc : développement du modèle M3S ».

Decaluwé, Bernard, André Martens, and Luc Savard (2001) *La politique économique du développement et les modèles d'équilibre général calculable*, Les Presses de l'Université de Montréal, Montréal.

Decaluwé, Bernard, Marie-Claude Martin, and Mokhtar Souissi (1993) *École PARADI de modélisation de politiques économiques de développement. Vol. 3 – Les modèles calculables d'équilibre général : les aspects financiers*, Université Laval, Québec.

Decaluwé, Bernard, and Mokhtar Souissi (1994) *Libéralisation financière en Tunisie : une étude rétrospective et prospective*, CRÉFA, Université Laval.

Fargeix, A., and E. Sadoulet (1994) « A Financial Computable General Equilibrium Model for the Analysis of Stabilisation Programs », Chapter 4 in Jean Mercenier and T. N. Srinivasan (1994) *Applied general equilibrium and economic development: present achievements and future trends*, University of Michigan Press.

Lemelin, André (2005) « La dette obligataire dans un MÉGC dynamique séquentiel », CIRPÉE (Centre Interuniversitaire sur le Risque, les Politiques Économiques et l'Emploi), Cahier de recherche 05-05, version révisée, mai.
http://132.203.59.36/CIRPEE/indexbase.htm
http://papers.ssrn.com/sol3/papers.cfm?abstract_id=690266

Rosensweig, Jeffrey A., and Lance Taylor (1990) « Devaluation, capital flows and crowding-out : A CGE model with portfolio choice for Thailand », Chapter 11 in Taylor, Lance (1990) *Socially relevant policy analysis : structuralist computable general equilibrium models for the developing world*, MIT Press, Cambridge, Mass.

Souissi, Mokhtar (1994) *Libéralisation financière, structure du capital et investissement: un MCEG avec actifs financiers appliqué à la Tunisie*, Doctoral dissertation, Université Laval, Québec.

Souissi, Mokhtar and Bernard Decaluwé (1997), « Financial deregulation in Tunisia : A prospective end retrospective analysis », CRÉFA, Université Laval, mai.

Thissen, Mark (1999) « Financial CGE models : Two decades of research », SOM research memorandum 99C02, SOM (Systems, Organizations and Management), Reijksuniversiteit Groningen, Groningen, juin.

APPENDIX 1 : INTEREST PAYABLE ON OUTSTANDING BONDS

On bonds issued in period t–1, the debtor will have to pay interests in the amount of

$$i^B_{t-1}\Delta B_{t-1} = \left[\left(1+i^B_{t-1}\right)-1\right]\Delta B_{t-1} = \left[\left(1+i^B_t\right)\frac{\left(1+i^B_{t-1}\right)}{\left(1+i^B_t\right)}-1\right]\Delta B_{t-1} = \left[\left(1+i^B_t\right)P^B_t-1\right]\Delta B_{t-1} \qquad [086]$$

where definition [006] is applied. On bonds issued in period t–2,

$$\begin{aligned} i^B_{t-2}\left(1-f_1\right)\Delta B_{t-2} &= \left[\left(1+i^B_{t-2}\right)-1\right]\left(1-f_1\right)\Delta B_{t-2} \\ &= \left[\left(1+i^B_{t-1}\right)\frac{\left(1+i^B_{t-2}\right)}{\left(1+i^B_{t-1}\right)}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\ &= \left[\left(1+i^B_{t-1}\right)P^B_{t-1}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\ &= \left[\left(1+i^B_t\right)\frac{\left(1+i^B_{t-1}\right)}{\left(1+i^B_t\right)}P^B_{t-1}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\ &= \left[\left(1+i^B_t\right)P^B_tP^B_{t-1}-1\right]\left(1-f_1\right)\Delta B_{t-2} \end{aligned} \qquad [087]$$

And on bonds issued in period t–3,

$$\begin{aligned} & i^B_{t-3}\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_{t-3}\right)-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_{t-2}\right)\frac{\left(1+i^B_{t-3}\right)}{\left(1+i^B_{t-2}\right)}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_{t-2}\right)P^B_{t-2}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_{t-1}\right)\frac{\left(1+i^B_{t-2}\right)}{\left(1+i^B_{t-1}\right)}P^B_{t-2}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_{t-1}\right)P^B_{t-1}P^B_{t-2}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_t\right)\frac{\left(1+i^B_{t-1}\right)}{\left(1+i^B_t\right)}P^B_{t-1}P^B_{t-2}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\ &= \left[\left(1+i^B_t\right)P^B_tP^B_{t-1}P^B_{t-2}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \end{aligned} \qquad [088]$$

Generally, interest payable on bonds issued in period $t–\theta$, for $\theta \leq M$, is

$$i^B_{t-\theta}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} = \left[\left(1+i^B_{t-\theta}\right)-1\right]\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [089]$$

$$i_{t-\theta}^{B}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta}=\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)-1\right]\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [090]$$

$$i_{t-\theta}^{B}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta}=\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta}-\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [091]$$

Altogether, interest payable is equal to

$$\begin{aligned} INT_t &= \sum_{\theta=1}^{M} i_{t-\theta}^{B}\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta} \\ &= \sum_{\theta=1}^{M}\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}-\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}\right] \end{aligned}, \text{ with } f_0=0 \qquad [013]$$

The amount DN_t of face-value debt (before redemption of debt coming to maturity in period t, and before new bonds issued in t) is :

$$DN_t=\sum_{\theta=1}^{M}\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}\text{ , avec } f_0=0 \qquad [014]$$

So [013] can be written as

$$INT_t=\sum_{\theta=1}^{M} i_{t-\theta}^{B}\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}=\sum_{\theta=1}^{M}\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}\right]-DN_t \qquad [015]$$

APPENDIX 2 : HOUSEHOLDS' WEALTH CONSTRAINT

Households' portfolio allocation problem is

$$\underset{A_t, B_t}{MAX}\ VC = \left(1+i_t^A\right)A_t^* + \left(1+i_t^B\right)B_t^* \qquad [021]$$

subject to

$$A_t^* + B_t^* = W_t \qquad [022]$$

$$W_t = A_w\left[\delta_A\left(A_t^*\right)^\beta + \delta_B\left(B_t^*\right)^\beta\right]^{\frac{1}{\beta}} \qquad [023]$$

with elasticity of transformation

$\tau = \dfrac{1}{1-\beta}$, varying from minus infinity to zero. And, correspondingly, $\beta = \dfrac{\tau - 1}{\tau}$, with β varying from one to infinity.

The first step is to solve the portfolio allocation problem taking into account only constraint [023] and temporarily ignoring constraint [022].

Solving the problem leads to demand functions

$$A_t = \frac{W_t}{A_w}\left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^\tau \left\{\delta_A{}^\tau\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^\tau\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [092]$$

and

$$B_t = \frac{W_t}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^\tau \left\{\delta_A{}^\tau\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^\tau\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [093]$$

(Demonstration given at the end of the appendix)

Replace A_t and B_t in [022] with their optimal value according to the demand functions and obtain

$$\begin{aligned} W_t = {} & \frac{W_t}{A_w}\left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^\tau \left\{\delta_A{}^\tau\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^\tau\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \\ & + \frac{W_t}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^\tau \left\{\delta_A{}^\tau\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^\tau\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \end{aligned} \qquad [094]$$

$$1 = \frac{1}{A_w}\left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^{\tau} \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}}$$
$$+ \frac{1}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^{\tau} \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [095]$$

$$A_w = \left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^{\tau} \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}}$$
$$+ \left(\frac{\delta_B}{1+i_t^B}\right)^{\tau} \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [096]$$

and, finally,

$$A_w = \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [097]$$

The asset demand functions can now be written as

$$A_t{}^{*} = W_t \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [026]$$

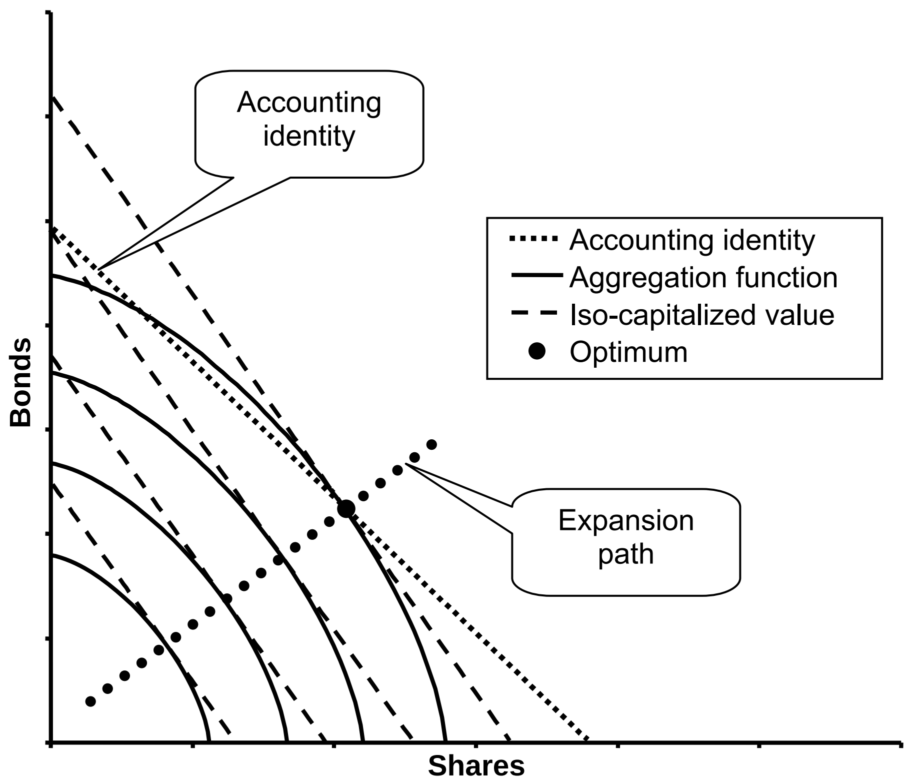
and

$$B_t{}^{*} = W_t \frac{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [027]$$

A_w is not a parameter : it is a variable whose value depends on rates of return and the price of shares. But that variable need not appear in the model : asset demand functions [026] and [027] automatically conform to wealth constraint [022].

Geometrically, A_w determines the location of constraint [023] in A_t B_t space. This is illustrated in Figure 1.

Figure 1 – Household portfolio allocation



Portfolio allocation equilibrium is located at the intersection of the expansion path and wealth accounting identity constraint [022]. The expansion path consists of the set of optimal asset combinations, for given rates of return and different values of CET aggregate [023]; for any optimal asset combination, the marginal rate of transformation of diversification constraint [023] is equal to the slope of the iso-capitalized value line (each one of which is given by asset combinations that yield a given value of utility *VC* [021]. In equilibrium, the value of A_w is the one that makes aggregation function [023] tangent to an iso-capitalized value line at its point of intersection with wealth constraint [022]. The slope of indifference curves [021] is $-\frac{\left(1+i_t^A\right)}{\left(1+i_t^B\right)_t}$.

The slope of [022] is equal to –1. So [022] intersects [023] from above or from below, according to whether $i_t^A >$ or $< i_t^B$. The case illustrated in Figure 1 is $i_t^A > i_t^B$.

Derivation of demand functions without constraint [022]

Form the Langrangian

$$\Lambda = \left(1+i_t^A\right)A_t{}^* + \left(1+i_t^B\right)B_t{}^* - \lambda\left[A_w\left[\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right]^{\frac{1}{\beta}} - W_t\right] \qquad [098]$$

First-order conditions are

$$\frac{\partial\Lambda}{\partial A_t{}^*} = \left(1+i_t^A\right) - \lambda\left[A_w \frac{1}{\beta}\left(\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right)^{\frac{1}{\beta}-1}\delta_A\beta\left(A_t{}^*\right)^{\beta-1}\right] = 0 \qquad [099]$$

$$\frac{\partial\Lambda}{\partial B_t{}^*} = \left(1+i_t^B\right) - \lambda\left[A_w \frac{1}{\beta}\left(\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right)^{\frac{1}{\beta}-1}\delta_B\beta\left(B_t{}^*\right)^{\beta-1}\right] = 0 \qquad [100]$$

Or

$$\frac{\partial\Lambda}{\partial A_t{}^*} = \left(1+i_t^A\right) - \lambda\left[A_w\left(\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right)^{\frac{1}{\beta}-1}\delta_A\left(A_t{}^*\right)^{\beta-1}\right] = 0 \qquad [101]$$

$$\frac{\partial\Lambda}{\partial B_t{}^*} = \left(1+i_t^B\right) - \lambda\left[A_w\left(\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right)^{\frac{1}{\beta}-1}\delta_B\left(B_t{}^*\right)^{\beta-1}\right] = 0 \qquad [102]$$

where constraint [023] is equivalent to

$$\left(\frac{W_t}{A_w}\right)^\beta = \left[\delta_A\left(A_t{}^*\right)^\beta + \delta_B\left(B_t{}^*\right)^\beta\right] \qquad [103]$$

so that

$$\frac{\partial\Lambda}{\partial A_t{}^*} = \left(1+i_t^A\right) - \lambda\left[A_w\left(\frac{W_t}{A_w}\right)^{1-\beta}\delta_A\left(A_t{}^*\right)^{\beta-1}\right] = 0 \qquad [104]$$

$$\frac{\partial\Lambda}{\partial B_t^*} = \left(1+i_t^B\right) - \lambda\left[A_w\left(\frac{W_t}{A_w}\right)^{1-\beta}\delta_B\left(B_t^*\right)^{\beta-1}\right] = 0 \qquad [105]$$

Develop

$$\left(1+i_t^A\right) = \lambda A_w\left(\frac{W_t}{A_w}\right)^{1-\beta}\delta_A\left(A_t^*\right)^{\beta-1} \qquad [106]$$

$$\left(1+i_t^B\right) = \lambda A_w\left(\frac{W_t}{A_w}\right)^{1-\beta}\delta_B\left(B_t^*\right)^{\beta-1} \qquad [107]$$

$$\left(A_t^*\right)^{\beta-1} = \frac{\left(1+i_t^A\right)}{\lambda\delta_A A_w}\left(\frac{W_t}{A_w}\right)^{\beta-1} \qquad [108]$$

$$\left(B_t^*\right)^{\beta-1} = \frac{\left(1+i_t^B\right)}{\lambda\delta_B A_w}\left(\frac{W_t}{A_w}\right)^{\beta-1} \qquad [109]$$

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\lambda\delta_A A_w}\right]^{\frac{1}{\beta-1}}\left(\frac{W_t}{A_w}\right) \qquad [110]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\lambda\delta_B A_w}\right]^{\frac{1}{\beta-1}}\left(\frac{W_t}{A_w}\right) \qquad [111]$$

Substitution into [023] yields

$$W_t = A_w\left[\delta_A\left(A_t^*\right)^{\beta} + \delta_B\left(B_t^*\right)^{\beta}\right]^{\frac{1}{\beta}} \qquad [023]$$

$$W_t = A_w\left\{\delta_A\left[\frac{\left(1+i_t^A\right)}{\lambda\delta_A A_w}\right]^{\frac{\beta}{\beta-1}}\left(\frac{W_t}{A_w}\right)^{\beta} + \delta_B\left[\frac{\left(1+i_t^B\right)}{\lambda\delta_B A_w}\right]^{\frac{\beta}{\beta-1}}\left(\frac{W_t}{A_w}\right)^{\beta}\right\}^{\frac{1}{\beta}} \qquad [112]$$

$$W_t = A_w W_t \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{1}{A_w} \right)^{\beta} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{1}{A_w} \right)^{\beta} \right\}^{\frac{1}{\beta}} \quad [113]$$

$$1 = A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A A_w} \left(\frac{1}{A_w} \right)^{\beta-1} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B A_w} \left(\frac{1}{A_w} \right)^{\beta-1} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} \quad [114]$$

$$A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A} \left(\frac{1}{A_w} \right)^{\beta} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B} \left(\frac{1}{A_w} \right)^{\beta} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \quad [115]$$

$$A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \left(A_w\right)^{\beta} \delta_A} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \left(A_w\right)^{\beta} \delta_B} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \quad [116]$$

$$A_w \left[\frac{1}{\lambda \left(A_w\right)^{\beta}} \right]^{\frac{1}{\beta-1}} \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\delta_A} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\delta_B} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \quad [117]$$

$$A_w \left[\frac{1}{\lambda \left(A_w\right)^{\beta}} \right]^{\frac{1}{\beta-1}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \quad [118]$$

$$\left(A_w\right)^{-\frac{1}{\beta-1}} \lambda^{-\frac{1}{\beta-1}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \quad [119]$$

$$\lambda^{\frac{1}{\beta-1}} = \left(A_w\right)^{\frac{1}{1-\beta}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \quad [120]$$

And the demand functions can be written

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A A_w}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right)\left(A_w\right)^{\frac{1}{1-\beta}} \left\{\left(\delta_A\right)^{-\frac{1}{\beta-1}}\left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}}\left(1+i_t^B\right)^{\frac{\beta}{\beta-1}}\right\}^{-\frac{1}{\beta}} \qquad [121]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B A_w}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right)\left(A_w\right)^{\frac{1}{1-\beta}} \left\{\left(\delta_A\right)^{-\frac{1}{\beta-1}}\left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}}\left(1+i_t^B\right)^{\frac{\beta}{\beta-1}}\right\}^{-\frac{1}{\beta}} \qquad [122]$$

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left\{\left(\delta_A\right)^{-\frac{1}{\beta-1}}\left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}}\left(1+i_t^B\right)^{\frac{\beta}{\beta-1}}\right\}^{-\frac{1}{\beta}} \qquad [123]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left\{\left(\delta_A\right)^{-\frac{1}{\beta-1}}\left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}}\left(1+i_t^B\right)^{\frac{\beta}{\beta-1}}\right\}^{-\frac{1}{\beta}} \qquad [124]$$

And, given

$$\tau = \frac{1}{1-\beta} \text{ and } \beta = \frac{\tau-1}{\tau} \qquad [024]$$

there results

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A}\right]^{-\tau} \left(\frac{W_t}{A_w}\right) \left\{\left(\delta_A\right)^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \left(\delta_B\right)^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [125]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B}\right]^{-\tau} \left(\frac{W_t}{A_w}\right) \left\{\left(\delta_A\right)^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \left(\delta_B\right)^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [126]$$

Q.E.D.

APPENDIX 3 : BUSINESS CAPITAL OWNERSHIP SHARES

We now develop [077], on the basis of which agent ownership fractions are updated between periods, according to each one's contribution to investments in the previous period. From [055], [077] becomes

$$\lambda^K_{j,t-1}\left(\frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}+\frac{SE_{t-1}}{IT_{t-1}}\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}\right)+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} \qquad [077]$$

$$\begin{aligned}&\lambda^K_{j,t-1}\frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}\\&\quad+\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}\end{aligned} \qquad [127]$$

$$\lambda^K_{j,t-1}\frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}}+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]\frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} \qquad [128]$$

$$\begin{aligned}&\lambda^K_{j,t-1}\left[1-\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\frac{Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}}\right]\\&\quad+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\frac{Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}}\end{aligned} \qquad [129]$$

$$\frac{\lambda^K_{j,t-1}\left[(1-\delta_i)KD_{i,t-1}+\left(1-\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}}$$

[130]

$$\frac{\lambda^K_{j,t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]Id_{i,t-1}}{\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}\left[(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}\right]}$$

[131]

$$\frac{\lambda^K_{j,t-1} PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right] + \left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]PK_{t-1}Id_{i,t-1}}{\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}PK_{t-1}\left[(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}\right]} \quad [132]$$

$$\frac{\lambda^K_{j,t-1} PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right] + \left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]PK_{t-1}Id_{i,t-1}}{PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+PK_{t-1}Id_{i,t-1}} \quad [133]$$

The ownership fraction of agent *j* in the capital of all industries (in the all-industries mutual fund), $\lambda^K_{j,t}$, is the weighted sum of industry fractions [131], with industry weights given by the denominator of [131]. So

$$\lambda^K_{j,t} = \frac{\lambda^K_{j,t-1} PK_{t-1}\sum_i\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right] + \left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]PK_{t-1}\sum_i Id_{i,t-1}}{PK_{t-1}\sum_i\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+PK_{t-1}\sum_i Id_{i,t-1}} \quad [134]$$

Substituting [050] and [076],

$$\lambda^K_{j,t} = \frac{\lambda^K_{j,t-1}WK_{i,t-1}+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]IT_{t-1}}{WK_{i,t-1}+IT_{t-1}} \quad [135]$$

$$\lambda^{K}_{j,t} = \frac{\lambda^{K}_{j,t-1} WK_{i,t-1} + \left[\lambda^{K}_{j,t-1} SE_{t-1} + \lambda^{I}_{j,t-1}\left(IT_{t-1} - SE_{t-1}\right)\right]}{WK_{i,t-1} + IT_{t-1}} \qquad [136]$$

$$\lambda^{K}_{j,t} = \frac{\lambda^{K}_{j,t-1}\left(WK_{i,t-1} + SE_{t-1}\right) + \lambda^{I}_{j,t-1}\left(IT_{t-1} - SE_{t-1}\right)}{WK_{i,t-1} + IT_{t-1}} \qquad [081]$$

APPENDIX 4 : NET EFFECT OF AN INCREASE IN MARKET RATE OF RETURN i_t^A ON HOUSEHOLD DEMAND FOR NEW SHARES

The demonstration is cumbersome, but quite straightforward. We proceed in four steps. We take the derivative, relative to i_t^A, first of the fraction of household wealth held in the form of shares, then of the overall value of the portfolio to be allocated, and finally, collecting results, of household demand for new shares.

A4.1 Effect on the fraction of household wealth held in the form of shares

According to [026], the fraction of household wealth held in the form of shares is

$$\frac{A_t^*}{W_t} = \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \quad [137]$$

Take the derivative with respect to i_t^A :

$$\frac{\partial}{\partial i_t^A}\left(\frac{A_t^*}{W_t}\right) = \frac{\left(\begin{array}{l}\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}\frac{\partial}{\partial i_t^A}\left[\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}\right] \\ -\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}\frac{\partial}{\partial i_t^A}\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}\end{array}\right)}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}^2} \quad [138]$$

$$\frac{\partial}{\partial i_t^A}\left(\frac{A_t^*}{W_t}\right) = \frac{\left(\begin{array}{l}\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}(-\tau)\left[\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau-1}\right] \\ -\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}(-\tau)\left[\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau-1}\right]\end{array}\right)}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}^2} \quad [139]$$

$$\frac{\partial}{\partial i_t^A}\left(\frac{A_t^*}{W_t}\right) = -\frac{\tau}{\left(1+i_t^A\right)}\frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}^2} \quad [140]$$

Since elasticity of transformation τ is negative, the above expression is clearly positive.

A2.2 Effect on the value of the portfolio to be allocated

Following [019], we have

$$\frac{\partial}{\partial i_t^A} W_t = \frac{\partial}{\partial i_t^A}\left[A_t + B_t + REMB_t + \sum_h SM_{h,t}\right] = \frac{\partial}{\partial i_t^A} A_t \qquad [141]$$

where, given [020],

$$\frac{\partial}{\partial i_t^A} A_t = \frac{\partial}{\partial i_t^A}\left(\lambda_{H,t}^K WK_t\right) = \lambda_{H,t}^K \frac{\partial}{\partial i_t^A} WK_t \qquad [142]$$

Now, substituting for $\zeta_{i,t}$ from [048] and [059], we rewrite [076] as

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)}\left(1 - \delta_i\right) KD_{i,t} + \left(\frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)} - 1\right) Id_{i,t}\right] \qquad [143]$$

$$WK_t = PK_t \left\{\sum_i \frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)}\left[\left(1 - \delta_i\right) KD_{i,t} + Id_{i,t}\right] - \sum_i Id_{i,t}\right\} \qquad [144]$$

So we have

$$\frac{\partial}{\partial i_t^A} WK_t = PK_t \sum_i \frac{\partial}{\partial i_t^A} \frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)}\left[\left(1 - \delta_i\right) KD_{i,t} + Id_{i,t}\right] \qquad [145]$$

$$\frac{\partial}{\partial i_t^A} WK_t = -PK_t \sum_i \frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)^2}\left[\left(1 - \delta_i\right) KD_{i,t} + Id_{i,t}\right] \qquad [146]$$

$$\frac{\partial}{\partial i_t^A} WK_t = -\frac{1}{\left(i_t^A + \delta_i\right)}\left(WK_t + \sum_i Id_{i,t}\right) \qquad [147]$$

Therefore,

$$\frac{\partial}{\partial i_t^A} W_t = \frac{\partial}{\partial i_t^A} A_t = \lambda_{H,t}^K \frac{\partial}{\partial i_t^A} WK_t = -\frac{\lambda_{H,t}^K}{\left(i_t^A + \delta_i\right)}\left(WK_t + \sum_i Id_{i,t}\right) \qquad [148]$$

A2.3 Household demand for new shares

Following [035], household demand for new shares is

$$\Delta A_t - CAB_t = W_t \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} - A_t \qquad [149]$$

Thus, the derivative of household demand for new shares is

$$\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = \frac{\partial}{\partial i_t^A} W_t \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} - \frac{\partial}{\partial i_t^A} A_t \qquad [150]$$

$$\begin{aligned}\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = {} & W_t \frac{\partial}{\partial i_t^A} \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \\ & + \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \frac{\partial}{\partial i_t^A} W_t - \frac{\partial}{\partial i_t^A} A_t\end{aligned} \qquad [151]$$

Collecting results from [140] and [148], we get

$$\begin{aligned}\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = {} & W_t \frac{(-\tau)}{\left(1+i_t^A\right)} \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}^2} \\ & + \left\{1 - \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left[\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right]}\right\} \frac{\lambda_{H,t}^K}{\left(i_t^A + \delta_i\right)}\left(WK_t + \sum_i Id_{i,t}\right)\end{aligned} \qquad [152]$$

Given that τ is negative, the first term on the right-hand side above is positive. And so is the second. The derivative of household demand for new shares relative to the market rate of return i_t^A is therefore positive.

APPPENDIX 5 : TECHNICAL DESCRIPTION OF EXTER-DEBT

The EXTER-Debt model was developed to demonstrate the feasibility of the modeling principle presented in this paper. It was constructed on the basis of the EXTER_DS[11] model, which itself derives from the EXTER[12] model.

A5.1 Model equations

A5.1.1 PRODUCTION

1. Value added production function of tradable good industry *j*

$$VA_j = A_j LD_j^{\alpha_j} KD_j^{1-\alpha_j}$$

```
VA(TR)   =E=  A(TR)*LD(TR)**alpha(TR)*KD(TR)**(1-alpha(TR));
```

2. Value added production function of non-tradable good industry *j*

$$VA_4 = LD_4$$

```
VA(NTR)  =E=  LD(NTR);
```

3. Industry *j* production (Leontief)

$$XS_j = \frac{VA_j}{v_j}$$

```
XS(I)    =E=  VA(I)/v(I);
```

4. Total intermediate consumption by industry *j* (Leontief)

$$CI_j = io_j XS_j$$

```
CI(I)    =E=  io(I)*XS(I);
```

5. Intermediate demand of good *i* by industry *j*

$$DI_{ij} = aij_{ij} CI_j$$

```
DI(TR,J) =E=  aij(TR,J)*CI(J);
```

---

11 Forthcoming.

12 Decaluwe, Martens, Savard, 2001. Details of the EXTER model are available on the *Poverty and Economic Policy* (PEP) website (http://www.pep-net.org/) : see the « Core Training Manuals on CGE Modeling » section in the MPIA training material, *Volume 2 – Basic CGE models (training material)*.

6. Total intermediate demand of good *i* by all industries

$$DIT_i = \sum_{j=1}^{4} aij_{ij} \, CI_j$$

```
DIT(TR)  =E=  SUM(J, DI(TR,J)) ;
```

7. Labor demand by tradable good industry *j*

$$LD_j = \frac{\alpha_j \, Pv_j \, VA_j}{w}$$

```
LD(TR)   =E=  PV(TR)*alpha(TR)*VA(TR)/w;
```

8. Labor demand by non-tradable good industry *j*

$$LD_4 = \frac{P_4 XS_4 - \sum_{i=1}^{3} Pc_i DI_{i4}}{w}$$

```
LD(NTR)  =E= (P(NTR)*XS(NTR)-SUM(TR,DI(TR,NTR)*PC(TR)))/w;
```

A5.1.2 INCOMES

9. Worker household income

$$YM_{hk} = w \, \lambda_h^L \sum_i LD_i + TGINT_{hk}$$

```
YH("HW") =E=  w*lam_L("HW")*SUM(I,LD(I)) + TGINT("HW");
```

10. Capitalist household income[13]

$$YM_{hk} = w \, \lambda_h^L \sum_i LD_i + \lambda_H^K DIV + TGINT_{hk}$$

```
YH("HC") =E=  w*lam_L("HC")*SUM(I,LD(I)) + lam_K("HH")*DIV + TGINT("HC");
```

11. Transfers to households, including interests on the public debt

$$TGINT_{h,t} = TG_{h,t} + \lambda_h^{PIN} INT_t \qquad [016]$$

```
TGINT(H)  =E= TG(H) + lam_PIN(H)*PINT;
```

12. Household disposable income

$$YDM_h = YM_h - TD_h$$

```
YDH(H)   =E=  YH(H) - TDH(H);
```

[13] In the present version, parameters `lambda` and `lambda_row` are zero.

13. Business income

$$YE = \sum_{j=1}^{3} r_j KD_j$$

```
YF        =E=  *SUM(TR,r(TR)*KD(TR));
```

14. Dividends paid by business

$$DIV = YE - TDE - SE$$

```
DIV       =E=  YF - TDF - SF ;
```

15. Household savings

$$SM_h = \psi_h YDM_h$$

```
SH(H)  =E=  psi(H)*YDH(H) ;
```

16. Business savings

$$SE = \psi_E (YE - TDE)$$

```
SAVF..            SF       =E=  psi_F*(YF - TDF) ;
```

17. Indirect tax revenue

$$TI_j = tx_j (P_j XS_j - Pe_j EX_j) + tx_j (1 + tm_j) e\, Pwm_j M_j$$

```
TI(TR)    =E=  tx(TR)*(P(TR)*XS(TR)-PE(TR)*EX(TR))+
               tx(TR)/(1+tx(TR))*PM(TR)*M(TR);
```

18. Import tax revenue

$$TIM_j = tm_j Pwm_j\ e\, M_j$$

```
TIM(TR)  =E=  tm(TR)*PWM(TR)*e*M(TR);
```

19. Export tax revenue

$$TIE_j = te_j Pe_j EX_j$$

```
TIE(TR)  =E=  te(TR)*PE(TR)*EX(TR);
```

20. Household income tax revenue

$$TD_h = ty_h YM_h$$

```
TDH(H)    =E=  tyh(H)*YH(H)  ;
```

21. Business income tax revenue

$$TDE = tye\, YE$$

```
TDF       =E=  tyf*YF  ;
```

22. Total government revenue

$$YG = \sum_{j=1}^{3} TIM_j + \sum_{j=1}^{3} TIE_j + \sum_{j=1}^{3} TI_j + \sum_{h=hs}^{hk} TD_h + TDE + \lambda_G^K \; DIV$$

```
YG        =E=  SUM(TR,TI(TR))+SUM(H,TDH(H))+SUM(TR,TIE(TR))
               + SUM(TR,TIM(TR)) + TDF + lam_K("G")*DIV   ;
```

23. Government savings

$$SG = YG - G - TG_{hs}$$

```
SG        =E=  YG - G - TG;
```

A5.1.3 GOOD SUPPLY AND DEMAND

24. CET aggregation of industry *j*'s products

$$XS_j = B_j^e \left[\beta_j^e EX_j^{-\kappa_j^e} + \left(1-\beta_j^e\right) D_j^{-\kappa_j^e} \right]^{-\frac{1}{\kappa_j^e}} \text{ où } \kappa_j^e = \frac{1-\tau_j^e}{\tau_j^e}$$

```
XS(TR)    =E=  B_E(TR)*(beta_e(TR)*EX(TR)**kappa_e(TR)
              +(1-beta_e(TR))*D(TR)**kappa_e(TR))
              **(1/kappa_e(TR));
```

with

```
kappa_e(TR)  = (1+tau_e(TR))/tau_e(TR);
```

where `tau_e(TR)` = $-\tau_j^e > 0$

so that `kappa_e(TR)` $= \dfrac{1+\left(-\tau_j^e\right)}{\left(-\tau_j^e\right)} = -\dfrac{1-\tau_j^e}{\tau_j^e} = -\kappa_j^e$

25. Supply of good *j* on the domestic market

$$D_j = \left[\left(\frac{1-\beta_j^e}{\beta_j^e} \right) \left(\frac{Pe_j}{Pl_j} \right) \right]^{\tau_j^e} EX_j \text{ or } EX_j = \left[\left(\frac{1-\beta_j^e}{\beta_j^e} \right) \left(\frac{Pe_j}{Pl_j} \right) \right]^{-\tau_j^e} D_j$$

```
EX(TR)    =E=  ((PE(TR)/PL(TR))**tau_e(TR)
               *((1-beta_e(TR))/beta_e(TR))**tau_e(TR))*D(TR);
```

where `tau_e(TR)` = $-\tau_j^e > 0$

26. Demand for composite good *j*

$$Q_j = A_j^m \left[\alpha_j^m M_j^{-\rho_j^m} + \left(1-\alpha_j^m\right) D_j^{-\rho_j^m} \right]^{-\frac{1}{\rho_j^m}} \text{ where } \rho_j^m = \frac{\sigma_j^m - 1}{\sigma_j^m}$$

```
Q(TR)    =E=  A_M(TR)*(alpha_m(TR)*M(TR)**(-rho_m(TR))
              +(1-alpha_m(TR))*D(TR)**(-rho_m(TR)))
              **(-1/rho_m(TR));
rho_m(TR)      = (1-sigma_m(TR))/sigma_m(TR);
```

where `sigma_m(TR)` = $\sigma_j^m > 0$

so that `rho_m(TR)` = $\frac{1-\sigma_j^m}{\sigma_j^m} = -\frac{\sigma_j^m - 1}{\sigma_j^m} = \rho_j^m$

27. Demand for imported good *j*

$$M_j = \left[\left(\frac{\alpha_j^m}{1-\alpha_j^m}\right)\left(\frac{Pd_j}{Pm_j}\right)\right]^{\sigma_j^m} D_j$$

```
M(TR)    =E=  ((alpha_m(TR)/(1-alpha_m(TR)))**(sigma_m(TR))
              *(PD(TR)/PM(TR))**sigma_m(TR))*D(TR);
```

where `sigma_m(TR)` = $\sigma_j^m > 0$

28. Consumption demand of good *i* by household *h*

$$C_{ih} = \frac{\gamma_{ih} YDM_h}{Pc_i}$$

```
C(TR,H)  =E=  gamma(TR,H)*YDH(H)/PC(TR) ;
```

29. Investment demand of good *i*

$$INV_i = \frac{\mu_i IT}{Pc_i}$$

```
INV(TR)  =E=  mu(TR)*IT/PC(TR) ;
```

30. Demand for non-tradable good

$$G = P_4 XS_4$$

```
G        =E=  XS("NTSER")* P("NTSER");
```

A5.1.4 INVESTMENT DEMAND BY INDUSTRY

31. Investment demand by industry *i* (Jung-Thorbecke)

$$\frac{Id_{i,t}}{KD_{i,t}} = \gamma 1_i \left(\frac{(1-tye)r_{i,t}}{U_{i,t}}\right)^{el\_ind_i} \qquad [046]$$

```
IND(tr)/KD(tr) =E=  g1(tr)*((1-tyf)*R(tr)/U(tr))**el_ind(tr);
```

32. User cost of capital in industry *i*

$$U_{i,t} = PK_t \zeta_{i,t} = PK_t\left(\phi_t + \delta_i\right) = PK_t\left(i_t^A + \delta_i\right) \qquad [060]$$

```
U(tr) =E= PK*(irac+delt(tr));
```

33. Total investment expenditures

$$IT_t = PK_t \sum_i Id_{i,t} \qquad [050]$$

```
IT      =E=   PK * SUM(tr,IND(tr));
```

A5.1.5 ASSET MARKETS

34. Price in period *t* of a bond issued in *t*–1, coming to maturity in *t*+1, and bearing interest at rate i_{t-1}^B

$$P_t^B = \frac{\left(1 + i_{t-1}^B\right)}{\left(1 + i_t^B\right)} \qquad [006]$$

```
POBT   =E= (1+iBEMI("t_1")) / (1+ir) ;
```

35. Amount of debt to be redeemend in period *t*

$$REMB_t = \sum_{\theta=1}^{M} \Delta B_{t-\theta}^{\theta} = \sum_{m=1}^{M} f_m \Delta B_{t-m} \qquad [012]$$

```
REMB    =E=   SUM(PAST,fech(PAST)*BEMIS(PAST)) ;
```

36. Interest payable on bonds in period *t*

$$INT_t = \sum_{\theta=1}^{M} i_{t-\theta}^B \left(1 - \sum_{m=0}^{\theta-1} f_m\right) \Delta B_{t-\theta}\text{, with } f_0 = 0 \qquad [013]$$

For *M* = 3 :

$$INT_t = i_{t-1}^B \Delta B_{t-1} + i_{t-2}^B\left(1 - f_1\right)\Delta B_{t-2} + i_{t-3}^B\left(1 - f_1 - f_2\right)\Delta B_{t-3}$$

```
PINT    =E=  iBEMI("t_1")*BEMIS("t_1")
             + iBEMI("t_2")*BEMIS("t_2")*(1-fech("t_1"))
             + iBEMI("t_3")*BEMIS("t_3")*(1-fech("t_1")-fech("t_2")) ;
```

See A5.4 for *M* = 10.

37. Public investments

$$PK_t IG_t = \pi_{IP} IT_t \qquad [029]$$

```
IPUB    =E=   part_ip * IT ;
```

38. Supply of new bonds

$$\Delta B_t = PK_t IG_t + REMB_t - SG_t \qquad [028]$$

```
NOVOB  =E=  IPUB + REMB - SG ;
```

39. Value of bonds outstanding from previous periods and not coming to maturity in the current period

$$B_t = \sum_{\theta=1}^{M-1} \sum_{\tau=1}^{M-\theta} \left(\prod_{s=1}^{\theta} P^B_{t-\theta+s} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta} \qquad [018]$$

For *M* = 3 :

$$B_t = \Delta B_t + \sum_{\theta=1}^{3-1} \sum_{\tau=1}^{3-\theta} \left(\prod_{s=1}^{\theta} P^B_{t-\theta+s} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta}$$

$$B_t = \Delta B_t + \sum_{\theta=1}^{2} \sum_{\tau=1}^{3-\theta} \left(\prod_{s=1}^{\theta} P^B_{t-\theta+s} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta}$$

$$B_t = \Delta B_t + \sum_{\tau=1}^{3-1} \left(\prod_{s=1}^{1} P^B_{t-1+s} \right)^{\tau} f_{1+\tau} \Delta B_{t-1} + \sum_{\tau=1}^{3-2} \left(\prod_{s=1}^{2} P^B_{t-2+s} \right)^{\tau} f_{2+\tau} \Delta B_{t-2}$$

$$B_t = \Delta B_t + \left(\prod_{s=1}^{1} P^B_{t-1+s} \right)^{1} f_{1+1} \Delta B_{t-1} + \left(\prod_{s=1}^{1} P^B_{t-1+s} \right)^{2} f_{1+2} \Delta B_{t-1}$$
$$+ \left(\prod_{s=1}^{2} P^B_{t-2+s} \right)^{1} f_{2+1} \Delta B_{t-2}$$

$$B_t = \Delta B_t + P^B_t f_2 \Delta B_{t-1} + \left(P^B_t\right)^2 f_3 \Delta B_{t-1} + \left(P^B_{t-1} P^B_t\right) f_3 \Delta B_{t-2}$$

```
OBLI   =E=  NOVOB + POBT*fech("t_2")*BEMIS("t_1")
            + (POBT**2)*fech("t_3")*BEMIS("t_1")
            + POBT*POB("t_1")*fech("t_3")*BEMIS("t_2") ;
```

See A5.4 for *M* = 10.

40. Supply of new shares

$$\Delta A_t = PK_t IE_t - SE_t \qquad [031]$$

$$PK_t IE_t = \left(1 - \pi_{IP}\right) IT_t \qquad [032]$$

$$PK_t IG_t = \pi_{IP} IT_t \qquad [029]$$

```
NOVACT  =E=  IT-IPUB-SF ;
```

41. Gross rate of return on capital in industry i, before depreciation, but net of taxes on capital income

$$\rho_{i,t} = (1 - tye)\frac{r_{i,t}}{PK_t} \qquad [047]$$

```
rho(TR)  =E= (1-tyf)*r(TR) / PK ;
```

42. Average net rate of return on capital (net of depreciation and taxes)

$$\overline{\rho} = \frac{\sum_j (\rho_j - \delta_j) KD_j}{\sum_j KD_j}$$

```
rmoy_K  =E=  SUM(TR,(rho(TR)-delt(TR))*KD(TR))/SUM(TR,KD(TR)) ;
```

43. Market value of shares inherited from the past

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\zeta_{i,t}} (1 - \delta_i) KD_{i,t} + \left(\frac{\rho_{i,t}}{\zeta_{i,t}} - 1 \right) Id_{i,t} \right] \qquad [076]$$

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)} (1 - \delta_i) KD_{i,t} + \left(\frac{\rho_{i,t}}{\left(i_t^A + \delta_i\right)} - 1 \right) Id_{i,t} \right] \qquad [143]$$

```
WK  =E=  PK*SUM(TR,rho(TR)*(1-delt(TR))*KD(TR)/(irac + delt(TR))
          - (1-rho(TR)/(irac + delt(TR)))*IND(TR) );
```

44. Market value of shares issued in previous periods held in the household's portfolio at the beginning of the period

$$A_t = \lambda_{H,t}^K WK_t \qquad [020]$$

```
VALACT =E= lam_K(HH) * WK ;
```

45. Value of the household portfolio to be allocated

$$W_t = A_t + B_t + REMB_t + \sum_h SM_{h,t} \qquad [019]$$

```
PTF  =E= VALACT + OBLI + REMB + SUM(H,SH(H));
```

46. Household demand for shares

$$A_t + \Delta A_t - CAB_t = W_t \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [035]$$

```
VALACT + NOVACT  =E=  CAB
                      + PTF*((d_portA/(1+irAC))**el_port)
                      /(((d_portA/(1+irAC))**el_port)
                        + (d_portB/(1+ir))**el_port) ;
```

47. Household demand for bonds

$$B_t + \Delta B_t = W_t \frac{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [037]$$

```
OBLI + NOVOB  =E=  PTF*((d_portB/(1+ir))**el_port)
                   /(((d_portA/(1+irAC))**el_port)
                     + (d_portB/(1+ir))**el_port) ;
```

A5.1.6 PRICES

48. Price of industry *j* value added

$$Pv_j = \frac{P_j XS_j - \sum_{i=1}^{3} Pc_i DI_{ij}}{VA_j}$$

```
PV(I)    =E= (P(I)*XS(I)-SUM(TR,DI(TR,I)*PC(TR)))/VA(I);
```

49. Rental rate of capital in industry *j*, before taxes and depreciation

$$r_j = \frac{Pv_j VA_j - w\,LD_j}{KD_j}$$

```
R(TR)    =E= (PV(TR)*VA(TR) - w*LD(TR) )/KD(TR) ;
```

50. Tax-inclusive price of imported good *j*

$$Pm_j = e\,Pwm_j\left(1+tm_j\right)\left(1+tx_j\right)$$

```
PM(TR)   =E=  (1+TX(TR))*(1+tm(TR))*e*PWM(TR);
```

51. Price received for exported good *j*

$$Pe_j = \frac{e\,Pwe_j}{\left(1+te_j\right)}$$

```
PE(TR)   =E=  PWE(TR)*e/(1+te(TR));
```

52. Domestic market price of good *j*, taxes included

$$Pc_j = \frac{Pd_j D_j + Pm_j M_j}{Q_j}$$

```
PC(TR)   =E=  (PD(TR)*D(TR)+PM(TR)*M(TR))/Q(TR);
```

53. Price of good *j* on the domestic market, before taxes

$$Pd_j = PL_j\left(1 + tx_j\right)$$

```
PD(TR)   =E=  PL(TR)*(1+tx(TR));
```

54. Producer price of good *j*

$$P_j = \frac{PL_j D_j + Pe_j EX_j}{XS_j}$$

```
P(TR)    =E=  (PL(TR)*D(TR) + PE(TR)*EX(TR))/XS(TR);
```

55. Value added Laspeyres price index

$$P_{index} = \sum_{j=1}^{4} Pv_j \delta_j$$

```
PINDEX   =E=  SUM(I,PV(I)*delta(I));
```

where

```
delta(I)  = PVO(I)*VAO(I)/SUM(J,PVO(J)*VAO(J));
```

56. Replacement price of capital

$$PK_t = \prod_i \left(\frac{P_{i,t}}{\mu_i}\right)^{\mu_i}$$

```
PK   =E= PROD(TR $ (INVO(TR) ne 0)  ,(PC(TR)/mu(TR))**mu(TR));
```

A5.1.7 EQUILIBRIUM

57. Labor market equilibrium

$$LS = \sum_{j=1}^{4} LD_j$$

```
LS      =E=   SUM(I,LD(I));
```

58. Domestic market equilibrium for good i

$$Q_i = DIT_i + \sum_{h=hs}^{hk} C_{ih} + INV_i$$

```
Q(GOOD) =E=   SUM(H,C(GOOD,H))+DIT(GOOD)+INV(GOOD);
LEON    =E=   Q("SER")- SUM(H,C("SER",H))-DIT("SER");
```

59. Investment-savings equilibrium

$$IT_t = \sum_h SM_{h,t} + SE_t + SG_t + CAB_t \qquad [038]$$

```
IT      =E=   SUM(H,SH(H)) + SF + SG + CAB;
```

60. Current account balance

$$CAB = e\sum_{j=1}^{3} Pwm_j M_j + \lambda_w \sum_{j=1}^{3} r_j KD_j + DIV\_ROW - e\sum_{j=1}^{3} Pwe_j EX_j$$

```
CAB      =E=  e*SUM(TR,PWM(TR)*M(TR)) - e*SUM(TR,PWE(TR)*EX(TR))
              + lam_K("RW")*DIV;
```

A5.1.8 DYNAMICS : BETWEEN-PERIODS VARIABLE UPDATING

61. Capital accumulation

$$KD_{i,t+1} = (1-\delta)KD_{i,t} + Id_{i,t}$$

```
KD.FX(TR) = (1-delt(tr))*KD.L(tr) + IND.L(TR);
```

62. Demographic growth

$$LS_{t+1} = (1+n)LS_t \qquad [082]$$

```
LS.FX = (1 + n)*LS.L ;
```

63. Evolution of public transfers to households

$$TG_{h,t+1} = (1+n)TG_{h,t} \qquad [083]$$

```
TG.FX(H) = (1+n)*TG.L(H) ;
```

64. Evolution of government spending

$$G_{t+1} = (1+n)G_t \qquad [084]$$

```
G.FX = (1+n)*G.L ;
```

65. Model closure and evolution of the current account balance

$$CAB_{t+1} = (1+n)CAB_t \qquad [085]$$

```
CAB.FX = (1+n)*CAB.L ;
```

66. Ownership shares

$$\lambda^{I}_{G,t-1} = \frac{\pi_{IP} IT_{t-1}}{IT_{t-1} - SE_{t-1}} \qquad [078]$$

$$\lambda^{I}_{RoW,t-1} = \frac{CAB_{t-1}}{IT_{t-1} - SE_{t-1}} \qquad [079]$$

$$\lambda^{I}_{H,t} = 1 - \lambda^{I}_{RoW,t} - \lambda^{I}_{G,t} \qquad [080]$$

$$\lambda^{K}_{j,t} = \frac{\lambda^{K}_{j,t-1}\left(WK_{i,t-1} + SE_{t-1}\right) + \lambda^{I}_{j,t-1}\left(IT_{t-1} - SE_{t-1}\right)}{WK_{i,t-1} + IT_{t-1}} \qquad [081]$$

```
lam_I.FX("G") = part_ip * IT.L / (IT.L - SF.L);
lam_I.FX("RW") = CAB.L / (IT.L - SF.L);
lam_I.FX("HH")  = 1 - lam_I.L("G") - lam_I.L("RW") ;
lam_K.FX(AG)  = (lam_K.L(AG)*(WK.L + SF.L) + lam_I.L(AG)*(IT.L - SF.L))
                 / (WK.L + IT.L);
```

A5.2 Model variables

Symbol	**GAMS acronym**	**Description**
A_t	**VALACT**	**Market value of shares held by households at beginning of period**
B_t	**OBLI**	**Value of bonds issued in previous periods and not coming to maturity in current period**
CAB	**CAB**	**Current account balance : deficit (–) or surplus (+)**
C_{ih}	**C(TR,H)**	**Consumption demand of product *i* by household *h***
CI_j	**CI(I)**	**Total intermediate consumption of industry *j***
DI_{ij}	**DI(TR,J)**	**Intermediate consumption of product *i* by industry *j***
DIT_i	**DIT(TR)**	**Total intermediate demand for product *i***
DIV	**DIV**	**Dividends paid by business**
D_j	**D(TR)**	**Domestic demand for domestic products**
e	**e**	**Exchange rate (price of foreign currency)**
EX_j	**EX(TR)**	**Exports**
G	**G**	**Total government expenditures**

i_t^A	`irac`	**Rate of return on shares**
i_t^B	`ir`	**Interest rate on government bonds (public debt)**
$i_{t-\theta}^B$	`iBEMI(PAST)`	**Interest rate on bonds issued in period $t - \theta$**
$Id_{i,t}$	`IND(tr)`	**Investment by destination industry**
INT_t	`PINT`	**Interest payments on bonds (public debt)**
INV_i	`INV(TR)`	**Investment demand for product i**
IT	`IT`	**Total investment expenditure**
KD_j	`KD(TR)`	**Industry demand for capital**
LD_j	`LD(I)`	**Industry demand for labor**
LS	`LS`	**Total labor supply**
M_j	`M(TR)`	**Imports**
Pc_j	`PC(TR)`	**Prices of composite goods**
Pd_j	`PD(TR)`	**Domestic prices of domestic products, taxes included**
Pe_j	`PE(TR)`	**Domestic price of exported goods**
P_{index}	`PINDEX`	**Value added price index**
P_j	`P(I)`	**Producer prices**
P_t^B	`POBT`	**Price of bonds issued in t–1, and coming to maturity in t+1**
$P_{t-\theta}^B$	`POB(PAST)`	**Price in period $t - \theta$ of a bond issued in period $t - \theta - 1$, with a maturity term of 2**
PK	`PK`	**Price (replacement cost) of capital**
$PK_t\ IG_t$	`IPUB`	**Public investment**
PL_j	`PL(TR)`	**Domestic prices of domestic products, net of taxes**
Pm_j	`PM(TR)`	**Domestic price of imported goods**
Pv_j	`PV(I)`	**Industry value added prices**
Pwe_j	`PWE(TR)`	**World price of exports, in foreith currency**

Pwm_j	PWM(TR)	**World price of imports, in foreign currency**
Q_j	Q(TR)	**Domestic demand for composite goods**
$REMB_t$	REMB	**Amount redemption of bonds coming to maturity in *t***
r_j	R(TR)	**Rental rate of capital in industry *j***
SE	SF	**Business savings**
SG	SG	**Government savings**
SM_h	SH(H)	**Household savings**
TDE	TDF	**Revenue from business income tax**
TD_h	TDH(H)	**Revenue from household income taxes**
TG_h	TG(H)	**Public transfers to households, except interests on public debt**
$TGINT_h$	TGINT(H)	**Public transfers to households, incl. interests on public debt**
TIE_j	TIE(TR)	**Revenue from export taxes**
TI_j	TI(TR)	**Revenue from indirect taxes**
TIM_j	TIM(TR)	**Revenue from import duties**
$U_{i,t}$	U(tr)	**User cost of capital**
VA_j	VA(I)	**Value added**
w	W	**Wage rate**
WK_t	WK	**Market value of capital inherited from the past**
W_t	PTF	**Value of household portfolio**
XS_j	XS(I)	**Production**
YDM_h	YDH(H)	**Household disposable incomes**
YE	YF	**Business income**
YG	YG	**Government revenue**
YM_h	YH(H)	**Household incomes**
ΔA_t	NOVACT	**Value of new shares issued in current period**

ΔB_t	**NOVOB**	**Value of new bonds issued in current period**
$\Delta B_{t-\theta}$	**BEMIS(PAST)**	**Bonds issued in period *t* – θ**
λ_{ag}^{I}	**lam_K(ag)**	**Agents' participation in last period's new share issue**
λ_{ag}^{K}	**lam_I(ag)**	**Capital ownership shares of agents**
$\bar{\rho}$	**rmoy_K**	**Average rate of return on capital**
ρ_i	**rho(tr)**	**Net-of-taxes rate of return on industry *i* capital**
	LEON	**Walras' Law checking variable**
	omega	**Objective function**

A5.3 Calibration for a regular path

This part of Appendix 5 details the calibration procedure applied in order to achieve the « quasi-regular » path discussed in section 5.

In what follows, the « O » appended to variable names designates base-year values.

A5.3.1 INVESTMENT DEMAND

Next, the stock of capital is calibrated as

$$KDO_i = \frac{IdO_i}{(\delta_i + n)} \quad [153]$$

```
KDO(TR)     = INDO(TR)/(delt(TR)+n);
```

Then the rental rate of capital is deduced from the amount gross capital income in the SAM :

$$rO_i = \frac{RKDO_i}{KDO_i} \quad [154]$$

```
ro(TR)    = RKDO(TR)/KDO(TR);
```

where $RKDO_i$ is capital income.

The gross (before depreciation) after-tax rate of return to capital is calibrated following equation [047] :

$$\rho_{i,t} = \frac{(1-tye)r_{i,t}}{PK_t} \quad [047]$$

```
rhoo(TR)  = (1 - tyf) * ro(TR) / PKO ;
```

And the average net rate of return on capital (after depreciation and taxes) is :

$$\overline{\rho} = \frac{\sum_j \left(\rho_j - \delta_j\right) KD_j}{\sum_j KD_j} \qquad [155]$$

```
rmoy_Ko      = SUM(TR,(rhoo(TR)-delt(TR))*KDO(TR))/SUM(TR,KDO(TR)) ;
```

The initial value of the rate of return on shares is simply set equal to $\overline{\rho}$:

$$i_0^A = \overline{\rho}_0$$

```
iraco        = rmoy_Ko ;
```

Consequently, according to [060], the user cost of capital is

$$U_{i,t} = PK_t \zeta_{i,t} = PK_t \left(\phi_t + \delta_i\right) = PK_t \left(i_t^A + \delta_i\right) \qquad [060]$$

```
UO(tr)       = PKO*(iraco+delt(tr));
```

Finally, given the exogenous elasticity parameter el_ind_i, the investment function constant is calibrated, by reversing [046] :

$$\gamma 1_i = \frac{Id_{i,t}}{KD_{i,t}} \left(\frac{(1-tye) r_{i,t}}{U_{i,t}} \right)^{-el\_ind_i} \qquad [156]$$

```
g1(tr) =  INDO(tr)/(KDO(tr)*((1-tyf)*RO(tr)/(UO(tr)))**el_ind(tr));
```

A5.3.2 BOND MARKET

First of all, past period bond interest rates are set equal to the current one.

```
iBEMIO(PAST)     = iro ;
```

The current price of a bond issued in t–1, coming to maturity in t+1, and bearing interest at rate i_{t-1}^B, is determined as in [006]:

```
POBTO            = (1+iBEMIO("t_1")) / (1+iro) ;
```

Past prices of two-period bonds are computed in the same way :

```
POBO("t_1")      = (1+iBEMIO("t_2")) / (1+iBEMIO("t_1")) ;
POBO("t_2")      = (1+iBEMIO("t_3")) / (1+iBEMIO("t_2")) ;
```

etc.

The amount of bonds issued in every past period is calibrated backwards from t–1 by applying the demographic growth rate :

```
BEMISO("t_2")     = BEMISO("t_1") / (1+n) ;
BEMISO("t_3")     = BEMISO("t_2") / (1+n) ;
```

etc.

The amount of bond redemption follows [012] :

```
REMBO           = SUM(PAST,fech(PAST)*BEMISO(PAST)) ;
```

Interests payable in the base period *PINTO* are according to [013]. Government savings can be calibrated only after *PINTO* has been computed, and initial government transfers to households calculated from [016] :

```
SGO            = YGO - GO - SUM(H,TGINTO(H)) ;
```

The amount of new bonds issued is calibrated so that the new issue has grown at a rate of n relative to the previous period's, in order to generate a quasi-regular path :

```
NOVOBO = (1+n)*BEMISO("t_1") ;
```

Consistent with [028], public investments are :

```
IPUBO = NOVOBO + SGO - REMBO;
```

And the share of public investments is simply

```
part_ip = IPUBO / ITO ;
```

Finally, the value of bonds outstanding from previous periods and not coming to maturity in the current period is determined from [018].

A5.3.3 OWNERSHIP SHARES

Ownership shares and participations in current investments are calibrated from previously determined values, according to equations [078], [079], [080] and [081]

A5.3.4 PORTFOLIO AND HOUSEHOLD ASSET DEMAND PARAMETERS

The value of shares issued in the past in the household portfolio follows [019] and [143] :

```
WKO  =  PKO*SUM(TR,rhoo(TR)*(1-delt(TR))*KDO(TR)/(iraco + delt(TR))
             - (1-rhoo(TR)/(iraco + delt(TR)))*INDO(TR) );
VALACTO          = lam_Ko("HH")*WKO ;
```

New shares issued are determined according to

```
NOVACTO         = ITO - IPUBO - SFO ;
```

And the value of the portfolio to be allocated is consistent with [019] :

```
PTFO            = VALACTO + OBLIO + REMBO + SUM(H,SHO(H));
```

Elasticity parameter τ is a free parameter, whose value is determined outside the model. And there are observed values of W_t, A_t, ΔA_t, B_t, ΔB_t, i_t^A, and i_t^B. So the calibration procedure concerns share parameters δ_A and δ_B.

Given

$$A_t + \Delta A_t - CAB_t = W_t \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [035]$$

$$B_t + \Delta B_t = W_t \frac{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \qquad [037]$$

there follows

$$\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t} = \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}} \qquad [157]$$

$$\left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t}\right)^{\frac{1}{\tau}} = \frac{\delta_A\left(1+i_t^A\right)^{-1}}{\delta_B\left(1+i_t^B\right)^{-1}} \qquad [158]$$

$$\frac{\delta_A}{\delta_B} = \left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t}\right)^{\frac{1}{\tau}} \frac{\left(1+i_t^A\right)}{\left(1+i_t^B\right)} \qquad [159]$$

Parameters δ_A and δ_B are defined only to a factor of proportionality. So one of them can be arbitrarily set to 1. Therefore

$$\delta_B = 1 \qquad [160]$$

$$\delta_A = \left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t}\right)^{\frac{1}{\tau}} \frac{\left(1+i_t^A\right)}{\left(1+i_t^B\right)} \qquad [161]$$

```
d_portA     = ((VALACTO+NOVACTO-CABO)/(OBLIO+NOVOBO))**(1/el_port)
              *(1+iraco)/(1+iro) ;
```

A5.4 Extension of calculations to *M* = 10

A5.4.1 INTEREST PAYABLE ON BONDS IN PERIOD *t*

The amount of interest paid on bonds is given by

$$\begin{aligned} INT_t &= \sum_{\theta=1}^{M} i_{t-\theta}^B \left(1 - \sum_{m=0}^{\theta-1} f_m\right) \Delta B_{t-\theta} \\ &= \sum_{\theta=1}^{M} \left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1} P_{t-s}^B\right)\left(1 - \sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} - \left(1 - \sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta}\right] \end{aligned} \qquad [013]$$

So, with M = 10,

```
PINT    =E= iBEMI("t_1")*BEMIS("t_1")
            + iBEMI("t_2")*BEMIS("t_2")*(1-fech("t_1"))
            + iBEMI("t_3")*BEMIS("t_3")*(1-fech("t_1")-fech("t_2"))
            + iBEMI("t_4")*BEMIS("t_4")
              *(1-fech("t_1")-fech("t_2")-fech("t_3"))
            + iBEMI("t_5")*BEMIS("t_5")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4"))
            + iBEMI("t_6")*BEMIS("t_6")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4")
                -fech("t_5"))
            + iBEMI("t_7")*BEMIS("t_7")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4")
                -fech("t_5")-fech("t_6"))
            + iBEMI("t_8")*BEMIS("t_8")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4")
                -fech("t_5")-fech("t_6")-fech("t_7"))
            + iBEMI("t_9")*BEMIS("t_9")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4")
                -fech("t_5")-fech("t_6")-fech("t_7")-fech("t_8"))
            + iBEMI("t_10")*BEMIS("t_10")
              *(1-fech("t_1")-fech("t_2")-fech("t_3")-fech("t_4")
                -fech("t_5")-fech("t_6")-fech("t_7")-fech("t_8")
                -fech("t_9")) ;
```

A5.4.2 VALUE OF BONDS OUTSTANDING FROM PREVIOUS PERIODS AND NOT COMING TO MATURITY IN CURRENT PERIOD

The value of bonds outstanding in period t, before redemption of bonds coming to maturity, but before the issue of new bonds, is

$$B_t = \sum_{\theta=1}^{M-1} \sum_{\tau=1}^{M-\theta} \left(\prod_{s=1}^{\theta} P_{t-\theta+s}^{B} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta} \qquad [018]$$

For M = 10,

$$B_t = \sum_{\theta=1}^{10-1} \sum_{\tau=1}^{10-\theta} \left(\prod_{s=1}^{\theta} P_{t-\theta+s}^{B} \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta} \qquad [162]$$

$$
\begin{aligned}
B_t = & \sum_{\tau=1}^{10-1}\left(\prod_{s=1}^{1} P_{t-1+s}^{B}\right)^{\tau} f_{1+\tau}\Delta B_{t-1} \\
& + \sum_{\tau=1}^{10-2}\left(\prod_{s=1}^{2} P_{t-2+s}^{B}\right)^{\tau} f_{2+\tau}\Delta B_{t-2} \\
& + \sum_{\tau=1}^{10-3}\left(\prod_{s=1}^{3} P_{t-3+s}^{B}\right)^{\tau} f_{3+\tau}\Delta B_{t-3} \\
& + \sum_{\tau=1}^{10-4}\left(\prod_{s=1}^{4} P_{t-4+s}^{B}\right)^{\tau} f_{4+\tau}\Delta B_{t-4} \\
& + \sum_{\tau=1}^{10-5}\left(\prod_{s=1}^{5} P_{t-5+s}^{B}\right)^{\tau} f_{5+\tau}\Delta B_{t-5} \\
& + \sum_{\tau=1}^{10-6}\left(\prod_{s=1}^{6} P_{t-6+s}^{B}\right)^{\tau} f_{6+\tau}\Delta B_{t-6} \\
& + \sum_{\tau=1}^{10-7}\left(\prod_{s=1}^{7} P_{t-7+s}^{B}\right)^{\tau} f_{7+\tau}\Delta B_{t-7} \\
& + \sum_{\tau=1}^{10-8}\left(\prod_{s=1}^{8} P_{t-8+s}^{B}\right)^{\tau} f_{8+\tau}\Delta B_{t-8} \qquad [163] \\
& + \sum_{\tau=1}^{10-9}\left(\prod_{s=1}^{9} P_{t-9+s}^{B}\right)^{\tau} f_{9+\tau}\Delta B_{t-9}
\end{aligned}
$$

$$
\begin{aligned}
B_t = &\sum_{\tau=1}^{9}\left(\prod_{s=1}^{1} P^B_{t-1+s}\right)^{\tau} f_{1+\tau}\Delta B_{t-1} \\
&+ \sum_{\tau=1}^{8}\left(\prod_{s=1}^{2} P^B_{t-2+s}\right)^{\tau} f_{2+\tau}\Delta B_{t-2} \\
&+ \sum_{\tau=1}^{7}\left(\prod_{s=1}^{3} P^B_{t-3+s}\right)^{\tau} f_{3+\tau}\Delta B_{t-3} \\
&+ \sum_{\tau=1}^{6}\left(\prod_{s=1}^{4} P^B_{t-4+s}\right)^{\tau} f_{4+\tau}\Delta B_{t-4} \\
&+ \sum_{\tau=1}^{5}\left(\prod_{s=1}^{5} P^B_{t-5+s}\right)^{\tau} f_{5+\tau}\Delta B_{t-5} \\
&+ \sum_{\tau=1}^{4}\left(\prod_{s=1}^{6} P^B_{t-6+s}\right)^{\tau} f_{6+\tau}\Delta B_{t-6} \\
&+ \sum_{\tau=1}^{3}\left(\prod_{s=1}^{7} P^B_{t-7+s}\right)^{\tau} f_{7+\tau}\Delta B_{t-7} \\
&+ \sum_{\tau=1}^{2}\left(\prod_{s=1}^{8} P^B_{t-8+s}\right)^{\tau} f_{8+\tau}\Delta B_{t-8} \qquad [164] \\
&+ \sum_{\tau=1}^{1}\left(\prod_{s=1}^{9} P^B_{t-9+s}\right)^{\tau} f_{9+\tau}\Delta B_{t-9}
\end{aligned}
$$

$$
\begin{aligned}
B_t &= \sum_{\tau=1}^{9}\left(P_t^B\right)^{\tau} f_{1+\tau}\Delta B_{t-1} \\
&+ \sum_{\tau=1}^{8}\left(P_t^B P_{t-1}^B\right)^{\tau} f_{2+\tau}\Delta B_{t-2} \\
&+ \sum_{\tau=1}^{7}\left(P_t^B P_{t-1}^B P_{t-2}^B\right)^{\tau} f_{3+\tau}\Delta B_{t-3} \\
&+ \sum_{\tau=1}^{6}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B\right)^{\tau} f_{4+\tau}\Delta B_{t-4} \\
&+ \sum_{\tau=1}^{5}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B\right)^{\tau} f_{5+\tau}\Delta B_{t-5} \\
&+ \sum_{\tau=1}^{4}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B\right)^{\tau} f_{6+\tau}\Delta B_{t-6} \\
&+ \sum_{\tau=1}^{3}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B\right)^{\tau} f_{7+\tau}\Delta B_{t-7} \\
&+ \sum_{\tau=1}^{2}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B P_{t-7}^B\right)^{\tau} f_{8+\tau}\Delta B_{t-8} \\
&+ \sum_{\tau=1}^{1}\left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B P_{t-7}^B P_{t-8}^B\right)^{\tau} f_{9+\tau}\Delta B_{t-9}
\end{aligned}
\qquad [165]
$$

```
OBLI   =E=  (POBT*fech("t_2")
             + (POBT**2)*fech("t_3")
             + (POBT**3)*fech("t_4")
             + (POBT**4)*fech("t_5")
             + (POBT**5)*fech("t_6")
             + (POBT**6)*fech("t_7")
             + (POBT**7)*fech("t_8")
             + (POBT**8)*fech("t_9")
             + (POBT**9)*fech("t_10")
                                      )*BEMIS("t_1")
      + ((POBT*POB("t_1"))*fech("t_3")
         + ((POBT*POB("t_1"))**2)*fech("t_4")
         + ((POBT*POB("t_1"))**3)*fech("t_5")
         + ((POBT*POB("t_1"))**4)*fech("t_6")
         + ((POBT*POB("t_1"))**5)*fech("t_7")
         + ((POBT*POB("t_1"))**6)*fech("t_8")
         + ((POBT*POB("t_1"))**7)*fech("t_9")
         + ((POBT*POB("t_1"))**8)*fech("t_10")
                                            )*BEMIS("t_2")
  + ((POBT*POB("t_1")*POB("t_2"))*fech("t_4")
     + ((POBT*POB("t_1")*POB("t_2"))**2)*fech("t_5")
     + ((POBT*POB("t_1")*POB("t_2"))**3)*fech("t_6")
     + ((POBT*POB("t_1")*POB("t_2"))**4)*fech("t_7")
     + ((POBT*POB("t_1")*POB("t_2"))**5)*fech("t_8")
     + ((POBT*POB("t_1")*POB("t_2"))**6)*fech("t_9")
     + ((POBT*POB("t_1")*POB("t_2"))**7)*fech("t_10")
```

```
                                        )*BEMIS("t_3")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))*fech("t_5")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))**2)*fech("t_6")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))**3)*fech("t_7")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))**4)*fech("t_8")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))**5)*fech("t_9")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3"))**6)*fech("t_10")
                                        )*BEMIS("t_4")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
    *POB("t_4"))*fech("t_6")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4"))**2)*fech("t_7")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4"))**3)*fech("t_8")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4"))**4)*fech("t_9")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4"))**5)*fech("t_10")
                                        )*BEMIS("t_5")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
    *POB("t_4")*POB("t_5"))*fech("t_7")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5"))**2)*fech("t_8")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5"))**3)*fech("t_9")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5"))**4)*fech("t_10")
                                        )*BEMIS("t_6")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
    *POB("t_4")*POB("t_5")*POB("t_6"))*fech("t_8")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5")*POB("t_6"))**2)*fech("t_9")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5")*POB("t_6"))**3)*fech("t_10")
                                        )*BEMIS("t_7")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
    *POB("t_4")*POB("t_5")*POB("t_6")*POB("t_7"))*fech("t_9")
   + ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
       *POB("t_4")*POB("t_5")*POB("t_6")*POB("t_7"))**2)*fech("t_10")
                                        )*BEMIS("t_8")
+ ((POBT*POB("t_1")*POB("t_2")*POB("t_3")
    *POB("t_4")*POB("t_5")*POB("t_6")*POB("t_7")*POB("t_8"))*fech("t_10")
                                        )*BEMIS("t_9") ;
```

LIST OF EQUATIONS

$$PO_t\left(\theta,\theta+\tau\right)=\frac{\left(1+i^B_{t-\theta}\right)^\tau}{\left(1+i^B_t\right)^\tau}$$ [001]

$$PO_t\ (1{,}2)=\frac{\left(1+i^B_{t-1}\right)}{\left(1+i^B_t\right)}$$ [002]

$$\left(1+i^B_{t-1}\right)=\left(1+i^B_t\right)PO_t\ (1{,}2)$$ [003]

$$\left(1+i^B_{t-\theta}\right)=\left(1+i^B_{t-\theta+1}\right)PO_{t-\theta+1}(1{,}2)=\left(1+i^B_{t-\theta+2}\right)PO_{t-\theta+2}(1{,}2)PO_{t-\theta+1}(1{,}2)=\cdots$$ [004]

$$\left(1+i^B_{t-\theta}\right)=\left(1+i^B_t\right)\prod_{s=1}^{\theta}PO_{t-\theta+s}(1{,}2)$$ [005]

$$P^B_t=PO_t\ (1{,}2)=\frac{\left(1+i^B_{t-1}\right)}{\left(1+i^B_t\right)}$$ [006]

$$\left(1+i^B_{t-\theta}\right)=\left(1+i^B_t\right)\prod_{s=1}^{\theta}P^B_{t-\theta+s}$$ [007]

$$PO_t\left(\theta,\theta+\tau\right)=\frac{\left(1+i^B_{t-\theta}\right)^\tau}{\left(1+i^B_t\right)^\tau}=\left(\prod_{s=1}^{\theta}P^B_{t-\theta+s}\right)^\tau$$ [008]

$$\sum_{m=1}^{M}f_m=1$$ [009]

$\Delta B^m_t=f_m\Delta B_t$, pour $m\leq M$ [010]

$$\sum_{m=1}^{M}\Delta B^m_t=\Delta B_t$$ [011]

$$REMB_t=\sum_{\theta=1}^{M}\Delta B^\theta_{t-\theta}=\sum_{m=1}^{M}f_m\Delta B_{t-m}$$ [012]

$$\begin{aligned}INT_t&=\sum_{\theta=1}^{M}i^B_{t-\theta}\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}\\&=\sum_{\theta=1}^{M}\left[\left(1+i^B_t\right)\left(\prod_{s=0}^{\theta-1}P^B_{t-s}\right)\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}-\left(1-\sum_{m=0}^{\theta-1}f_m\right)\Delta B_{t-\theta}\right]\end{aligned}$$ [013]

$$DN_t = \sum_{\theta=1}^{M}\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} \quad [014]$$

$$INT_t = \sum_{\theta=1}^{M} i_{t-\theta}^{B}\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta} = \sum_{\theta=1}^{M}\left[\left(1+i_t^B\right)\left(\prod_{s=0}^{\theta-1} P_{t-s}^{B}\right)\left(1-\sum_{m=0}^{\theta-1} f_m\right)\Delta B_{t-\theta}\right] - DN_t \quad [015]$$

$$TGINT_{h,t} = TG_{h,t} + \lambda_h^{PIN} INT_t \quad [016]$$

$$B_t = \sum_{\theta=1}^{M-1}\sum_{\tau=1}^{M-\theta} PO_t\,(\theta,\theta+\tau)\Delta B_{t-\theta}^{\theta+\tau} \quad [017]$$

$$B_t = \sum_{\theta=1}^{M-1}\sum_{\tau=1}^{M-\theta}\left(\prod_{s=1}^{\theta} P_{t-\theta+s}^{B}\right)^{\tau} f_{\theta+\tau}\Delta B_{t-\theta} \quad [018]$$

$$W_t = A_t + B_t + REMB_t + \sum_h SM_{h,t} \quad [019]$$

$$A_t = \lambda_{H,t}^{K} WK_t \quad [020]$$

$$\underset{A_t,B_t}{MAX}\; VC = \left(1+i_t^A\right)A_t^{*} + \left(1+i_t^B\right)B_t^{*} \quad [021]$$

$$A_t^{*} + B_t^{*} = W_t \quad [022]$$

$$W_t = A_w\left[\delta_A\left(A_t^{*}\right)^{\beta} + \delta_B\left(B_t^{*}\right)^{\beta}\right]^{\frac{1}{\beta}} \quad [023]$$

$$\tau = \frac{1}{1-\beta}\ \ (\beta > 1) \quad [024]$$

$$A_w = \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \quad [025]$$

$$A_t^{*} = W_t\,\frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \quad [026]$$

$$B_t^{*} = W_t\,\frac{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \quad [027]$$

$$\Delta B_t = PK_t IG_t + REMB_t - SG_t \quad [028]$$

$$PK_t IG_t = \pi_{IP} IT_t \qquad [029]$$

$$\Delta B_t = \pi_{IP} IT_t + REMB_t - SG_t \qquad [030]$$

$$\Delta A_t = PK_t IE_t - SE_t \qquad [031]$$

$$PK_t IE_t = (1 - \pi_{IP}) IT_t \qquad [032]$$

$$\Delta A_t = (1 - \pi_{IP}) IT_t - SE_t \qquad [033]$$

$$A_t^* = A_t + \Delta A_t - CAB_t \qquad [034]$$

$$A_t + \Delta A_t - CAB_t = W_t \frac{\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau}\right\}} \qquad [035]$$

$$B_t^* = B_t + \Delta B_t \qquad [036]$$

$$B_t + \Delta B_t = W_t \frac{\delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau}\right\}} \qquad [037]$$

$$IT_t = \sum_h SM_{h,t} + SE_t + SG_t + CAB_t \qquad [038]$$

$$B_t + \Delta B_t + A_t + \Delta A_t - CAB_t = W_t \qquad [039]$$

$$B_t + \Delta B_t + A_t + \Delta A_t - CAB_t = W_t = A_t + B_t + REMB_t + \sum_h SM_{h,t} \qquad [040]$$

$$\Delta B_t + \Delta A_t - REMB_t = CAB_t + \sum_h SM_{h,t} \qquad [041]$$

$$\left[\pi_{IP} IT_t + REMB_t - SG_t\right] + \left[(1 - \pi_{IP}) IT_t - SE_t\right] - REMB_t = CAB_t + \sum_h SM_{h,t} \qquad [042]$$

$$\frac{1}{1 - \delta_i} \sum_{\theta=1}^{\infty} \left(\frac{1 - \delta_i}{1 + \phi_t}\right)^{\theta} (1 - tye) r_{i,t} = \frac{(1 - tye) r_{i,t}}{\phi_t + \delta_i} \qquad [043]$$

$$U_{i,t} = PK_t \left(\phi_t + \delta_i\right) \qquad [044]$$

$$\frac{(1 - tye) r_{i,t}}{PK_t \left(\phi_t + \delta_i\right)} = \frac{(1 - tye) r_{i,t}}{U_{i,t}} \qquad [045]$$

$$\frac{Id_{i,t}}{KD_{i,t}} = \gamma 1_i \left(\frac{(1 - tye) r_{i,t}}{U_{i,t}}\right)^{el\_ind_i} \qquad [046]$$

$$\rho_{i,t} = \frac{(1-tye)r_{i,t}}{PK_t} \quad [047]$$

$$\zeta_{i,t} = \frac{U_{i,t}}{PK_t} = \phi_t + \delta_i = \frac{1}{\frac{1}{1-\delta_i}\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}} = \frac{1-\delta_i}{\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}} \quad [048]$$

$$\frac{Id_{i,t}}{KD_{i,t}} = \gamma 1_i \left(\frac{(1-tye)r_{i,t}}{U_{i,t}}\right)^{el\_ind_i} = \gamma 1_i \left(\frac{\rho_{i,t}}{\zeta_{i,t}}\right)^{el\_ind_i} \quad [049]$$

$$IT_t = PK_t \sum_i Id_{i,t} \quad [050]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}}(1-tye)r_{i,t}\ KD_{i,t+1} = \zeta_{i,t}\ PK_t\ Id_{i,t} \quad [051]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}}(1-tye)r_{i,t}\ \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] = \zeta_{i,t}\ PK_t\ Id_{i,t} \quad [052]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}}\rho_{i,t}PK_t\ \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] = \zeta_{i,t}\ PK_t\ Id_{i,t} \quad [053]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}}\rho_{i,t}\ \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] = \zeta_{i,t}\ Id_{i,t} \quad [054]$$

$$\frac{\Delta N_{i,t}}{N_{i,t} + \Delta N_{i,t}} = \frac{\zeta_{i,t}}{\rho_{i,t}}\ \frac{Id_{i,t}}{(1-\delta_i)KD_{i,t} + Id_{i,t}} \quad [055]$$

$$\sum_{\theta=1}^{\infty}\frac{1}{(1+\phi_t)^{\theta}}i_t^A = \sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+\phi_t}\right)^{\theta}\zeta_{i,t} = 1 \quad [056]$$

$$\sum_{\theta=1}^{\infty}\frac{1}{(1+\phi_t)^{\theta}} = \frac{1}{\phi_t} \quad [057]$$

$$\sum_{\theta=1}^{\infty}\frac{1}{(1+\phi_t)^{\theta}}i_t^A = \frac{1}{\phi_t}i_t^A = 1 \quad [058]$$

$$i_t^A = \phi_t. \quad [059]$$

$$U_{i,t} = PK_t\zeta_{i,t} = PK_t(\phi_t + \delta_i) = PK_t(i_t^A + \delta_i) \quad [060]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} \quad [061]$$

$$\rho_{i,t}PK_t\ \left[(1-\delta_i)KD_{i,t}+Id_{i,t}\right]-\zeta_{i,t}\ PK_t\ Id_{i,t}=PK_t\left\{\rho_{i,t}\ \left[(1-\delta_i)KD_{i,t}+Id_{i,t}\right]-\zeta_{i,t}\ Id_{i,t}\right\} \quad [062]$$

$$\rho_{i,t}PK_t\ \left[(1-\delta_i)KD_{i,t}+Id_{i,t}\right]-\zeta_{i,t}\ PK_t\ Id_{i,t}=PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t}+\left(\rho_{i,t}-\zeta_{i,t}\right)Id_{i,t}\right] \quad [063]$$

$$\frac{1}{(1-\delta_i)}\sum_{\theta=1}^{\infty}\left(\frac{1-\delta_i}{1+i_t^A}\right)^{\theta}PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t}+\left(\rho_{i,t}-\zeta_{i,t}\right)Id_{i,t}\right]$$

$$=\frac{1}{i_t^A+\delta_i}PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t}+\left(\rho_{i,t}-\zeta_{i,t}\right)Id_{i,t}\right] \quad [064]$$

$$=\frac{1}{\zeta_{i,t}}PK_t\left[\rho_{i,t}(1-\delta_i)KD_{i,t}+\left(\rho_{i,t}-\zeta_{i,t}\right)Id_{i,t}\right] \quad [065]$$

$$=PK_t\left[\frac{\rho_{i,t}}{\zeta_{i,t}}(1-\delta_i)KD_{i,t}+\left(\frac{\rho_{i,t}}{\zeta_{i,t}}-1\right)Id_{i,t}\right] \quad [066]$$

$$=PK_t\left[\frac{\rho_{i,t}}{\zeta_{i,t}}KD_{i,t+1}-Id_{i,t}\right] \quad [067]$$

$$\Delta N_{i,t}=\frac{\zeta_{i,t}}{\rho_{i,t}}\frac{Id_{i,t}}{(1-\delta_i)KD_{i,t}+Id_{i,t}}\left(N_{i,t}+\Delta N_{i,t}\right) \quad [068]$$

$$\left(1-\frac{\zeta_{i,t}}{\rho_{i,t}}\frac{Id_{i,t}}{(1-\delta_i)KD_{i,t}+Id_{i,t}}\right)\Delta N_{i,t}=\frac{\zeta_{i,t}}{\rho_{i,t}}\frac{Id_{i,t}}{(1-\delta_i)KD_{i,t}+Id_{i,t}}N_{i,t} \quad [069]$$

$$\Delta N_{i,t}=\frac{\left(\dfrac{\zeta_{i,t}}{\rho_{i,t}}\dfrac{Id_{i,t}}{(1-\delta_i)KD_{i,t}+Id_{i,t}}\right)}{\left(1-\dfrac{\zeta_{i,t}}{\rho_{i,t}}\dfrac{Id_{i,t}}{(1-\delta_i)KD_{i,t}+Id_{i,t}}\right)}N_{i,t} \quad [070]$$

$$\Delta N_{i,t}=\frac{1}{\left(\dfrac{\rho_{i,t}}{\zeta_{i,t}}\dfrac{(1-\delta_i)KD_{i,t}+Id_{i,t}}{Id_{i,t}}-1\right)}N_{i,t} \quad [071]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t\ Id_{i,t}}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} \frac{(1-\delta_i)KD_{i,t} + Id_{i,t}}{Id_{i,t}} - 1 \right) \qquad [072]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} \left[(1-\delta_i)KD_{i,t} + Id_{i,t}\right] - Id_{i,t} \right) \qquad [073]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} = \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right) \qquad [074]$$

$$\frac{PK_t\ Id_{i,t}}{\Delta N_{i,t}} \Delta N_{i,t} + \frac{PK_t}{N_{i,t}} \left(\frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} - Id_{i,t} \right) N_{i,t} = PK_t \frac{\rho_{i,t}}{\zeta_{i,t}} KD_{i,t+1} \qquad [075]$$

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\zeta_{i,t}} (1-\delta_i) KD_{i,t} + \left(\frac{\rho_{i,t}}{\zeta_{i,t}} - 1 \right) Id_{i,t} \right] \qquad [076]$$

$$\lambda^K_{j,t-1} \left(\frac{N_{i,t-1}}{N_{i,t-1} + \Delta N_{i,t-1}} + \frac{SE_{t-1}}{IT_{t-1}} \frac{\Delta N_{i,t-1}}{N_{i,t-1} + \Delta N_{i,t-1}} \right) + \lambda^I_{j,t-1} \left(1 - \frac{SE_{t-1}}{IT_{t-1}} \right) \frac{\Delta N_{i,t-1}}{N_{i,t-1} + \Delta N_{i,t-1}} \qquad [077]$$

$$\lambda^I_{G,t-1} = \frac{\pi_{IP} IT_{t-1}}{IT_{t-1} - SE_{t-1}} \qquad [078]$$

$$\lambda^I_{RoW,t-1} = \frac{CAB_{t-1}}{IT_{t-1} - SE_{t-1}} \qquad [079]$$

$$\lambda^I_{H,t} = 1 - \lambda^I_{RoW,t} - \lambda^I_{G,t} \qquad [080]$$

$$\lambda^K_{j,t} = \frac{\lambda^K_{j,t-1} \left(WK_{i,t-1} + SE_{t-1} \right) + \lambda^I_{j,t-1} \left(IT_{t-1} - SE_{t-1} \right)}{WK_{i,t-1} + IT_{t-1}} \qquad [081]$$

$$LS_{t+1} = (1+n) LS_t \qquad [082]$$

$$TG_{h,t+1} = (1+n) TG_{h,t} \qquad [083]$$

$$G_{t+1} = (1+n) G_t \qquad [084]$$

$$CAB_{t+1} = (1+n) CAB_t \qquad [085]$$

$$i^B_{t-1} \Delta B_{t-1} = \left[\left(1 + i^B_{t-1} \right) - 1 \right] \Delta B_{t-1} = \left[\left(1 + i^B_t \right) \frac{\left(1 + i^B_{t-1} \right)}{\left(1 + i^B_t \right)} - 1 \right] \Delta B_{t-1} = \left[\left(1 + i^B_t \right) P^B_t - 1 \right] \Delta B_{t-1} \qquad [086]$$

$$\begin{aligned}
i_{t-2}^{B}\left(1-f_1\right)\Delta B_{t-2} &= \left[\left(1+i_{t-2}^{B}\right)-1\right]\left(1-f_1\right)\Delta B_{t-2} \\
&= \left[\left(1+i_{t-1}^{B}\right)\frac{\left(1+i_{t-2}^{B}\right)}{\left(1+i_{t-1}^{B}\right)}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\
&= \left[\left(1+i_{t-1}^{B}\right)P_{t-1}^{B}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\
&= \left[\left(1+i_{t}^{B}\right)\frac{\left(1+i_{t-1}^{B}\right)}{\left(1+i_{t}^{B}\right)}P_{t-1}^{B}-1\right]\left(1-f_1\right)\Delta B_{t-2} \\
&= \left[\left(1+i_{t}^{B}\right)P_{t}^{B}P_{t-1}^{B}-1\right]\left(1-f_1\right)\Delta B_{t-2}
\end{aligned} \qquad [087]$$

$$\begin{aligned}
& i_{t-3}^{B}\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t-3}^{B}\right)-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t-2}^{B}\right)\frac{\left(1+i_{t-3}^{B}\right)}{\left(1+i_{t-2}^{B}\right)}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t-2}^{B}\right)P_{t-2}^{B}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t-1}^{B}\right)\frac{\left(1+i_{t-2}^{B}\right)}{\left(1+i_{t-1}^{B}\right)}P_{t-2}^{B}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t-1}^{B}\right)P_{t-1}^{B}P_{t-2}^{B}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t}^{B}\right)\frac{\left(1+i_{t-1}^{B}\right)}{\left(1+i_{t}^{B}\right)}P_{t-1}^{B}P_{t-2}^{B}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3} \\
&= \left[\left(1+i_{t}^{B}\right)P_{t}^{B}P_{t-1}^{B}P_{t-2}^{B}-1\right]\left(1-f_1-f_2\right)\Delta B_{t-3}
\end{aligned} \qquad [088]$$

$$i_{t-\theta}^{B}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} = \left[\left(1+i_{t-\theta}^{B}\right)-1\right]\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [089]$$

$$i_{t-\theta}^{B}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} = \left[\left(1+i_{t}^{B}\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)-1\right]\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [090]$$

$$i_{t-\theta}^{B}\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} = \left(1+i_{t}^{B}\right)\left(\prod_{s=0}^{\theta-1}P_{t-s}^{B}\right)\left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} - \left(1-\sum_{m=1}^{\theta-1}f_m\right)\Delta B_{t-\theta} \qquad [091]$$

$$A_t = \frac{W_t}{A_w}\left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau}+\delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [092]$$

$$B_t = \frac{W_t}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau}+\delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \qquad [093]$$

$$W_t = \frac{W_t}{A_w}\left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}}$$
$$+\frac{W_t}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \quad [094]$$

$$1 = P_t^A\frac{1}{A_w}\left(\frac{\delta_A}{P_t^A\left(1+i_t^A\right)}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[P_t^A\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}}$$
$$+\frac{1}{A_w}\left(\frac{\delta_B}{1+i_t^B}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[P_t^A\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \quad [095]$$

$$A_w = \left(\frac{\delta_A}{\left(1+i_t^A\right)}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}}$$
$$+\left(\frac{\delta_B}{1+i_t^B}\right)^{\tau}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \quad [096]$$

$$A_w = \left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}\left\{\delta_A{}^{\tau}\left[\left(1+i_t^A\right)\right]^{1-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{1-\tau}\right\}^{\frac{\tau}{1-\tau}} \quad [097]$$

$$\Lambda = \left(1+i_t^A\right)A_t{}^* + \left(1+i_t^B\right)B_t{}^* - \lambda\left[A_w\left[\delta_A\left(A_t{}^*\right)^{\beta} + \delta_B\left(B_t{}^*\right)^{\beta}\right]^{\frac{1}{\beta}} - W_t\right] \quad [098]$$

$$\frac{\partial\Lambda}{\partial A_t{}^*} = \left(1+i_t^A\right) - \lambda\left[A_w\frac{1}{\beta}\left(\delta_A\left(A_t{}^*\right)^{\beta} + \delta_B\left(B_t{}^*\right)^{\beta}\right)^{\frac{1}{\beta}-1}\delta_A\beta\left(A_t{}^*\right)^{\beta-1}\right] = 0 \quad [099]$$

$$\frac{\partial\Lambda}{\partial B_t{}^*} = \left(1+i_t^B\right) - \lambda\left[A_w\frac{1}{\beta}\left(\delta_A\left(A_t{}^*\right)^{\beta} + \delta_B\left(B_t{}^*\right)^{\beta}\right)^{\frac{1}{\beta}-1}\delta_B\beta\left(B_t{}^*\right)^{\beta-1}\right] = 0 \quad [100]$$

$$\frac{\partial\Lambda}{\partial A_t{}^*} = \left(1+i_t^A\right) - \lambda\left[A_w\left(\delta_A\left(A_t{}^*\right)^{\beta} + \delta_B\left(B_t{}^*\right)^{\beta}\right)^{\frac{1}{\beta}-1}\delta_A\left(A_t{}^*\right)^{\beta-1}\right] = 0 \quad [101]$$

$$\frac{\partial \Lambda}{\partial B_t^*} = \left(1 + i_t^B\right) - \lambda \left[A_w \left(\delta_A \left(A_t^*\right)^{\beta} + \delta_B \left(B_t^*\right)^{\beta} \right)^{\frac{1}{\beta}-1} \delta_B \left(B_t^*\right)^{\beta-1} \right] = 0 \qquad [102]$$

$$\left(\frac{W_t}{A_w}\right)^{\beta} = \left[\delta_A \left(A_t^*\right)^{\beta} + \delta_B \left(B_t^*\right)^{\beta} \right] \qquad [103]$$

$$\frac{\partial \Lambda}{\partial A_t^*} = \left(1 + i_t^A\right) - \lambda \left[A_w \left(\frac{W_t}{A_w}\right)^{1-\beta} \delta_A \left(A_t^*\right)^{\beta-1} \right] = 0 \qquad [104]$$

$$\frac{\partial \Lambda}{\partial B_t^*} = \left(1 + i_t^B\right) - \lambda \left[A_w \left(\frac{W_t}{A_w}\right)^{1-\beta} \delta_B \left(B_t^*\right)^{\beta-1} \right] = 0 \qquad [105]$$

$$\left(1 + i_t^A\right) = \lambda A_w \left(\frac{W_t}{A_w}\right)^{1-\beta} \delta_A \left(A_t^*\right)^{\beta-1} \qquad [106]$$

$$\left(1 + i_t^B\right) = \lambda A_w \left(\frac{W_t}{A_w}\right)^{1-\beta} \delta_B \left(B_t^*\right)^{\beta-1} \qquad [107]$$

$$\left(A_t^*\right)^{\beta-1} = \frac{\left(1 + i_t^A\right)}{\lambda \delta_A A_w} \left(\frac{W_t}{A_w}\right)^{\beta-1} \qquad [108]$$

$$\left(B_t^*\right)^{\beta-1} = \frac{\left(1 + i_t^B\right)}{\lambda \delta_B A_w} \left(\frac{W_t}{A_w}\right)^{\beta-1} \qquad [109]$$

$$A_t^* = \left[\frac{\left(1 + i_t^A\right)}{\lambda \delta_A A_w} \right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \qquad [110]$$

$$B_t^* = \left[\frac{\left(1 + i_t^B\right)}{\lambda \delta_B A_w} \right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \qquad [111]$$

$$W_t = A_w \left\{ \delta_A \left[\frac{\left(1 + i_t^A\right)}{\lambda \delta_A A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{W_t}{A_w}\right)^{\beta} + \delta_B \left[\frac{\left(1 + i_t^B\right)}{\lambda \delta_B A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{W_t}{A_w}\right)^{\beta} \right\}^{\frac{1}{\beta}} \qquad [112]$$

$$W_t = A_w W_t \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{1}{A_w} \right)^{\beta} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B A_w} \right]^{\frac{\beta}{\beta-1}} \left(\frac{1}{A_w} \right)^{\beta} \right\}^{\frac{1}{\beta}} \tag{113}$$

$$1 = A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A A_w} \left(\frac{1}{A_w} \right)^{\beta-1} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B A_w} \left(\frac{1}{A_w} \right)^{\beta-1} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} \tag{114}$$

$$A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \delta_A} \left(\frac{1}{A_w} \right)^{\beta} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \delta_B} \left(\frac{1}{A_w} \right)^{\beta} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \tag{115}$$

$$A_w \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\lambda \left(A_w\right)^{\beta} \delta_A} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\lambda \left(A_w\right)^{\beta} \delta_B} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \tag{116}$$

$$A_w \left[\frac{1}{\lambda \left(A_w\right)^{\beta}} \right]^{\frac{1}{\beta-1}} \left\{ \delta_A \left[\frac{\left(1+i_t^A\right)}{\delta_A} \right]^{\frac{\beta}{\beta-1}} + \delta_B \left[\frac{\left(1+i_t^B\right)}{\delta_B} \right]^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \tag{117}$$

$$A_w \left[\frac{1}{\lambda \left(A_w\right)^{\beta}} \right]^{\frac{1}{\beta-1}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \tag{118}$$

$$\left(A_w\right)^{-\frac{1}{\beta-1}} \lambda^{-\frac{1}{\beta-1}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{\frac{1}{\beta}} = 1 \tag{119}$$

$$\lambda^{\frac{1}{\beta-1}} = \left(A_w\right)^{\frac{1}{1-\beta}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \tag{120}$$

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A A_w}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left(A_w\right)^{\frac{1}{1-\beta}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \quad [121]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B A_w}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left(A_w\right)^{\frac{1}{1-\beta}} \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \quad [122]$$

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \quad [123]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B}\right]^{\frac{1}{\beta-1}} \left(\frac{W_t}{A_w}\right) \left\{ \left(\delta_A\right)^{-\frac{1}{\beta-1}} \left[\left(1+i_t^A\right)\right]^{\frac{\beta}{\beta-1}} + \left(\delta_B\right)^{-\frac{1}{\beta-1}} \left(1+i_t^B\right)^{\frac{\beta}{\beta-1}} \right\}^{-\frac{1}{\beta}} \quad [124]$$

$$A_t^* = \left[\frac{\left(1+i_t^A\right)}{\delta_A}\right]^{-\tau} \left(\frac{W_t}{A_w}\right) \left\{ \left(\delta_A\right)^{\tau} \left[\left(1+i_t^A\right)\right]^{1-\tau} + \left(\delta_B\right)^{\tau} \left(1+i_t^B\right)^{1-\tau} \right\}^{\frac{\tau}{1-\tau}} \quad [125]$$

$$B_t^* = \left[\frac{\left(1+i_t^B\right)}{\delta_B}\right]^{-\tau} \left(\frac{W_t}{A_w}\right) \left\{ \left(\delta_A\right)^{\tau} \left[\left(1+i_t^A\right)\right]^{1-\tau} + \left(\delta_B\right)^{\tau} \left(1+i_t^B\right)^{1-\tau} \right\}^{\frac{\tau}{1-\tau}} \quad [126]$$

$$\lambda_{j,t-1}^K \frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} + \lambda_{j,t-1}^K \frac{SE_{t-1}}{IT_{t-1}} \frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} + \lambda_{j,t-1}^I \left(1-\frac{SE_{t-1}}{IT_{t-1}}\right) \frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} \quad [127]$$

$$\lambda_{j,t-1}^K \frac{N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} + \left[\lambda_{j,t-1}^K \frac{SE_{t-1}}{IT_{t-1}} + \lambda_{j,t-1}^I \left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right] \frac{\Delta N_{i,t-1}}{N_{i,t-1}+\Delta N_{i,t-1}} \quad [128]$$

$$\lambda^K_{j,t-1}\left[1-\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\frac{Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}}\right] + \left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\frac{Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}} \quad [129]$$

$$\frac{\lambda^K_{j,t-1}\left[(1-\delta_i)KD_{i,t-1}+\left(1-\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]\frac{\zeta_{i,t-1}}{\rho_{i,t-1}}Id_{i,t-1}}{(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}} \quad [130]$$

$$\frac{\lambda^K_{j,t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]Id_{i,t-1}}{\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}\left[(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}\right]} \quad [131]$$

$$\frac{\lambda^K_{j,t-1}PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]PK_{t-1}Id_{i,t-1}}{\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}PK_{t-1}\left[(1-\delta_i)KD_{i,t-1}+Id_{i,t-1}\right]} \quad [132]$$

$$\frac{\lambda^K_{j,t-1}PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+\left[\lambda^K_{j,t-1}\frac{SE_{t-1}}{IT_{t-1}}+\lambda^I_{j,t-1}\left(1-\frac{SE_{t-1}}{IT_{t-1}}\right)\right]PK_{t-1}Id_{i,t-1}}{PK_{t-1}\left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}(1-\delta_i)KD_{i,t-1}+\left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}}-1\right)Id_{i,t-1}\right]+PK_{t-1}Id_{i,t-1}} \quad [133]$$

$$\lambda_{j,t}^{K} = \frac{\lambda_{j,t-1}^{K} PK_{t-1} \sum_{i} \left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}} (1-\delta_{i}) KD_{i,t-1} + \left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}} - 1 \right) Id_{i,t-1} \right] + \left[\lambda_{j,t-1}^{K} \frac{SE_{t-1}}{IT_{t-1}} + \lambda_{j,t-1}^{I} \left(1 - \frac{SE_{t-1}}{IT_{t-1}} \right) \right] PK_{t-1} \sum_{i} Id_{i,t-1}}{PK_{t-1} \sum_{i} \left[\frac{\rho_{i,t-1}}{\zeta_{i,t-1}} (1-\delta_{i}) KD_{i,t-1} + \left(\frac{\rho_{i,t-1}}{\zeta_{i,t-1}} - 1 \right) Id_{i,t-1} \right] + PK_{t-1} \sum_{i} Id_{i,t-1}} \quad [134]$$

$$\lambda_{j,t}^{K} = \frac{\lambda_{j,t-1}^{K} WK_{i,t-1} + \left[\lambda_{j,t-1}^{K} \frac{SE_{t-1}}{IT_{t-1}} + \lambda_{j,t-1}^{I} \left(1 - \frac{SE_{t-1}}{IT_{t-1}} \right) \right] IT_{t-1}}{WK_{i,t-1} + IT_{t-1}} \quad [135]$$

$$\lambda_{j,t}^{K} = \frac{\lambda_{j,t-1}^{K} WK_{i,t-1} + \left[\lambda_{j,t-1}^{K} SE_{t-1} + \lambda_{j,t-1}^{I} (IT_{t-1} - SE_{t-1}) \right]}{WK_{i,t-1} + IT_{t-1}} \quad [136]$$

$$\frac{A_t^*}{W_t} = \frac{\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau}}{\left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\}} \quad [137]$$

$$\frac{\partial}{\partial i_t^A} \left(\frac{A_t^*}{W_t} \right) = \frac{\left(\begin{array}{l} \left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\} \frac{\partial}{\partial i_t^A} \left[\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} \right] \\ - \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} \frac{\partial}{\partial i_t^A} \left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\} \end{array} \right)}{\left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\}^2} \quad [138]$$

$$\frac{\partial}{\partial i_t^A} \left(\frac{A_t^*}{W_t} \right) = \frac{\left(\begin{array}{l} \left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\} (-\tau) \left[\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau-1} \right] \\ - \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} (-\tau) \left[\delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau-1} \right] \end{array} \right)}{\left\{ \delta_A{}^{\tau} \left(1 + i_t^A\right)^{-\tau} + \delta_B{}^{\tau} \left(1 + i_t^B\right)^{-\tau} \right\}^2} \quad [139]$$

$$\frac{\partial}{\partial i_t^A}\left(\frac{A_t^*}{W_t}\right) = -\frac{\tau}{\left(1+i_t^A\right)} \frac{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}\delta_B{}^\tau\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}+\delta_B{}^\tau\left(1+i_t^B\right)^{-\tau}\right\}^2} \quad [140]$$

$$\frac{\partial}{\partial i_t^A} W_t = \frac{\partial}{\partial i_t^A}\left[A_t + B_t + REMB_t + \sum_h SM_{h,t}\right] = \frac{\partial}{\partial i_t^A} A_t \quad [141]$$

$$\frac{\partial}{\partial i_t^A} A_t = \frac{\partial}{\partial i_t^A}\left(\lambda_{H,t}^K WK_t\right) = \lambda_{H,t}^K \frac{\partial}{\partial i_t^A} WK_t \quad [142]$$

$$WK_t = PK_t \sum_i \left[\frac{\rho_{i,t}}{\left(i_t^A+\delta_i\right)}\left(1-\delta_i\right)KD_{i,t} + \left(\frac{\rho_{i,t}}{\left(i_t^A+\delta_i\right)}-1\right) Id_{i,t}\right] \quad [143]$$

$$WK_t = PK_t \left\{\sum_i \frac{\rho_{i,t}}{\left(i_t^A+\delta_i\right)}\left[\left(1-\delta_i\right)KD_{i,t} + Id_{i,t}\right] - \sum_i Id_{i,t}\right\} \quad [144]$$

$$\frac{\partial}{\partial i_t^A} WK_t = PK_t \sum_i \frac{\partial}{\partial i_t^A}\frac{\rho_{i,t}}{\left(i_t^A+\delta_i\right)}\left[\left(1-\delta_i\right)KD_{i,t} + Id_{i,t}\right] \quad [145]$$

$$\frac{\partial}{\partial i_t^A} WK_t = -PK_t \sum_i \frac{\rho_{i,t}}{\left(i_t^A+\delta_i\right)^2}\left[\left(1-\delta_i\right)KD_{i,t} + Id_{i,t}\right] \quad [146]$$

$$\frac{\partial}{\partial i_t^A} WK_t = -\frac{1}{\left(i_t^A+\delta_i\right)}\left(WK_t + \sum_i Id_{i,t}\right) \quad [147]$$

$$\frac{\partial}{\partial i_t^A} W_t = \lambda_{H,t}^K \frac{\partial}{\partial i_t^A} WK_t = -\frac{\lambda_{H,t}^K}{\left(i_t^A+\delta_i\right)}\left(WK_t + \sum_i Id_{i,t}\right) \quad [148]$$

$$\Delta A_t - CAB_t = W_t \frac{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}+\delta_B{}^\tau\left(1+i_t^B\right)^{-\tau}\right\}} - A_t \quad [149]$$

$$\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = \frac{\partial}{\partial i_t^A} W_t \frac{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^\tau\left(1+i_t^A\right)^{-\tau}+\delta_B{}^\tau\left(1+i_t^B\right)^{-\tau}\right\}} - \frac{\partial}{\partial i_t^A} A_t \quad [150]$$

$$\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = W_t \frac{\partial}{\partial i_t^A} \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} + \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}} \frac{\partial}{\partial i_t^A} W_t - \frac{\partial}{\partial i_t^A} A_t \quad [151]$$

$$\frac{\partial}{\partial i_t^A}\left(\Delta A_t - CAB_t\right) = W_t \frac{(-\tau)}{\left(1+i_t^A\right)} \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}}{\left\{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right\}^2} + \left\{1 - \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\left[\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau} + \delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}\right]}\right\} \frac{\lambda_{H,t}^K}{\left(i_t^A + \delta_i\right)} \left(WK_t + \sum_i Id_{i,t}\right) \quad [152]$$

$$KDO_i = \frac{IdO_i}{\left(\delta_i + n\right)} \quad [153]$$

$$rO_i = \frac{RKDO_i}{KDO_i} \quad [154]$$

$$\bar{\rho} = \frac{\sum_j \left(\rho_j - \delta_j\right) KD_j}{\sum_j KD_j} \quad [155]$$

$$\gamma 1_i = \frac{Id_{i,t}}{KD_{i,t}} \left(\frac{(1-tye) r_{i,t}}{U_{i,t}}\right)^{-el\_ind_i} \quad [156]$$

$$\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t} = \frac{\delta_A{}^{\tau}\left(1+i_t^A\right)^{-\tau}}{\delta_B{}^{\tau}\left(1+i_t^B\right)^{-\tau}} \quad [157]$$

$$\left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t}\right)^{\frac{1}{\tau}} = \frac{\delta_A\left(1+i_t^A\right)^{-1}}{\delta_B\left(1+i_t^B\right)^{-1}} \quad [158]$$

$$\frac{\delta_A}{\delta_B} = \left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t}\right)^{\frac{1}{\tau}} \frac{\left(1+i_t^A\right)}{\left(1+i_t^B\right)} \quad [159]$$

$$\delta_B = 1 \quad [160]$$

$$\delta_A = \left(\frac{A_t + \Delta A_t - CAB_t}{B_t + \Delta B_t} \right)^{\frac{1}{\tau}} \frac{\left(1 + i_t^A\right)}{\left(1 + i_t^B\right)} \tag{161}$$

$$B_t = \sum_{\theta=1}^{10-1} \sum_{\tau=1}^{10-\theta} \left(\prod_{s=1}^{\theta} P_{t-\theta+s}^B \right)^{\tau} f_{\theta+\tau} \Delta B_{t-\theta} \tag{162}$$

$$\begin{aligned}
B_t = & \sum_{\tau=1}^{10-1} \left(\prod_{s=1}^{1} P_{t-1+s}^B \right)^{\tau} f_{1+\tau} \Delta B_{t-1} \\
& + \sum_{\tau=1}^{10-2} \left(\prod_{s=1}^{2} P_{t-2+s}^B \right)^{\tau} f_{2+\tau} \Delta B_{t-2} \\
& + \sum_{\tau=1}^{10-3} \left(\prod_{s=1}^{3} P_{t-3+s}^B \right)^{\tau} f_{3+\tau} \Delta B_{t-3} \\
& + \sum_{\tau=1}^{10-4} \left(\prod_{s=1}^{4} P_{t-4+s}^B \right)^{\tau} f_{4+\tau} \Delta B_{t-4} \\
& + \sum_{\tau=1}^{10-5} \left(\prod_{s=1}^{5} P_{t-5+s}^B \right)^{\tau} f_{5+\tau} \Delta B_{t-5} \\
& + \sum_{\tau=1}^{10-6} \left(\prod_{s=1}^{6} P_{t-6+s}^B \right)^{\tau} f_{6+\tau} \Delta B_{t-6} \\
& + \sum_{\tau=1}^{10-7} \left(\prod_{s=1}^{7} P_{t-7+s}^B \right)^{\tau} f_{7+\tau} \Delta B_{t-7} \\
& + \sum_{\tau=1}^{10-8} \left(\prod_{s=1}^{8} P_{t-8+s}^B \right)^{\tau} f_{8+\tau} \Delta B_{t-8} \\
& + \sum_{\tau=1}^{10-9} \left(\prod_{s=1}^{9} P_{t-9+s}^B \right)^{\tau} f_{9+\tau} \Delta B_{t-9}
\end{aligned} \tag{163}$$

$$
\begin{aligned}
B_t = &\sum_{\tau=1}^{9}\left(\prod_{s=1}^{1} P^{B}_{t-1+s}\right)^{\tau} f_{1+\tau}\Delta B_{t-1} \\
&+ \sum_{\tau=1}^{8}\left(\prod_{s=1}^{2} P^{B}_{t-2+s}\right)^{\tau} f_{2+\tau}\Delta B_{t-2} \\
&+ \sum_{\tau=1}^{7}\left(\prod_{s=1}^{3} P^{B}_{t-3+s}\right)^{\tau} f_{3+\tau}\Delta B_{t-3} \\
&+ \sum_{\tau=1}^{6}\left(\prod_{s=1}^{4} P^{B}_{t-4+s}\right)^{\tau} f_{4+\tau}\Delta B_{t-4} \\
&+ \sum_{\tau=1}^{5}\left(\prod_{s=1}^{5} P^{B}_{t-5+s}\right)^{\tau} f_{5+\tau}\Delta B_{t-5} \\
&+ \sum_{\tau=1}^{4}\left(\prod_{s=1}^{6} P^{B}_{t-6+s}\right)^{\tau} f_{6+\tau}\Delta B_{t-6} \\
&+ \sum_{\tau=1}^{3}\left(\prod_{s=1}^{7} P^{B}_{t-7+s}\right)^{\tau} f_{7+\tau}\Delta B_{t-7} \\
&+ \sum_{\tau=1}^{2}\left(\prod_{s=1}^{8} P^{B}_{t-8+s}\right)^{\tau} f_{8+\tau}\Delta B_{t-8} \qquad [164] \\
&+ \sum_{\tau=1}^{1}\left(\prod_{s=1}^{9} P^{B}_{t-9+s}\right)^{\tau} f_{9+\tau}\Delta B_{t-9}
\end{aligned}
$$

$$
\begin{aligned}
B_t &= \sum_{\tau=1}^{9} \left(P_t^B\right)^{\tau} f_{1+\tau} \Delta B_{t-1} \\
&+ \sum_{\tau=1}^{8} \left(P_t^B P_{t-1}^B\right)^{\tau} f_{2+\tau} \Delta B_{t-2} \\
&+ \sum_{\tau=1}^{7} \left(P_t^B P_{t-1}^B P_{t-2}^B\right)^{\tau} f_{3+\tau} \Delta B_{t-3} \\
&+ \sum_{\tau=1}^{6} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B\right)^{\tau} f_{4+\tau} \Delta B_{t-4} \\
&+ \sum_{\tau=1}^{5} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B\right)^{\tau} f_{5+\tau} \Delta B_{t-5} \\
&+ \sum_{\tau=1}^{4} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B\right)^{\tau} f_{6+\tau} \Delta B_{t-6} \\
&+ \sum_{\tau=1}^{3} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B\right)^{\tau} f_{7+\tau} \Delta B_{t-7} \\
&+ \sum_{\tau=1}^{2} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B P_{t-7}^B\right)^{\tau} f_{8+\tau} \Delta B_{t-8} \\
&+ \sum_{\tau=1}^{1} \left(P_t^B P_{t-1}^B P_{t-2}^B P_{t-3}^B P_{t-4}^B P_{t-5}^B P_{t-6}^B P_{t-7}^B P_{t-8}^B\right)^{\tau} f_{9+\tau} \Delta B_{t-9}
\end{aligned} \qquad [165]
$$