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## **Holdups and Overinvestment in Physical Capital Markets**

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**Abstract:**

Firms in many situations must make investment decisions long before they meet with new capital suppliers. In addition, most physical capital is specific to a task or location, thus implying potentially important switching costs in case negotiations between a firm and a supplier break down. The present paper analyzes the implications of these frictions. The sequentiality of investment makes it impossible to write binding ex-ante contracts. Together with the rents arising from switching costs, this implies a holdup problem. In partial equilibrium, firms react strategically by overinvesting so as to reduce their marginal productivity and thus the price of capital they negotiate with their suppliers upon matching. In general equilibrium, the holdup problem interacts with externalities from switching costs, resulting in inefficient allocations. In a more general macroeconomic context, the holdup problem in physical capital markets interacts with holdup problems in labor markets that typically lead to underinvestment. As long as capital and labor are complements, this presents the firm with a trade-off between overinvestment and overemployment that neutralizes, at least partially, the distortionary effects of each of the two holdup problems.

**Keywords:** Holdup problems, Trading frictions, Investment, Strategic bargaining

**JEL Classification:** D23, D24, E22

# 1 Introduction

Factor specificity and contract incompleteness are pervasive phenomena in economic transactions and considered to be a major source of inefficiency. Whenever one party expends resources that increase the value of a productive relationship relative to its outside options (i.e. specificity) and other participating parties can appropriate some of the quasi-rents arising from this investment (i.e. contract incompleteness), a holdup problem occurs. In partial equilibrium, holdup problems typically reduce the incentive to invest (e.g. Simons, 1944; or Grou, 1984). In general equilibrium, markets react such as to alleviate holdup problems, resulting in underutilization of resources, missing technology adoption, and excessive destruction (e.g. Caballero and Hammour, 1998a).

In this paper, I examine the consequences of holdup problems in physical capital markets and show that the frictions usually associated with underinvestment – specificity and contract incompleteness – can lead to exactly the opposite outcome: overinvestment. The result arises naturally in an environment where firms need to make investment decisions before they can participate in physical capital markets to find new suppliers; and where trading frictions make it costly to switch from one capital supplier to another. The first assumption implies that firms cannot enter into ex-ante contracts (explicit or implicit) with suppliers, simply because firms do not know their trading partners at the time of the investment decision. The second assumption implies that once matched, capital suppliers and firms enjoy quasi-rents. In conjunction with the inexistence of ex-ante contracts, this implies a holdup problem. As long as different capital goods are substitutes, the firm has a strategic incentive to accumulate ex-ante more capital than is socially optimal as this decreases the marginal productivity of new capital and thus the price over which the firm negotiates ex-post with its suppliers. In principle, capital suppliers would like to pursue exactly the inverse strategy. But since capital accumulation is a sequential process involving many different capital suppliers, each supplier takes the firm’s capital stock and consequently the price of capital as exogenous. Holdup problems in physical capital markets thus lead to overinvestment. This result is important because the policy implications of overinvestment (e.g. for optimal capital taxation) are radically different than for underinvestment, which is the result typically emphasized by the holdup literature.

The overinvestment result crucially hinges on the presence of ex-ante investment decisions by firms and switching costs in physical capital markets. Neither of these assumptions seems very

restrictive. Capital accumulation is an inherently sequential process, which means that a large fraction of the firm's capital stock is predetermined when the firm meets with new capital suppliers. Furthermore, firms often need to plan for new ventures and secure financing long before they look for suppliers. Switching costs, in turn, are a natural implication of the fact that most physical capital is specific to a particular task and/or location. Classical time-to-build and adjustment cost models of investment implicitly build on the same assumption, but keep the price of capital determined competitively. By contrast, the present paper posits that specificity gives rise to trading frictions that render physical capital markets decentralized and leave prices open to bilateral bargaining.

After a review of the existing literature on holdup problems in Section 2, Section 3 derives the overinvestment result in a basic model with homogenous capital as the only factor of production. To make the concept of specificity and trading frictions explicit, I assume as in Kurmann and Petrosky-Nadeau (2007) that the allocation of capital from suppliers to firms is subject to random matching.<sup>1</sup> Once matched, capital suppliers bargain with their respective firm over the price of capital. Following Stole and Zwiebel (1996a), I assume that at any time before production starts, a capital supplier may enter into pairwise renegotiation with its firm. Likewise, a firm may call in any supplier for renegotiation. Within each pairwise negotiation, the firm and the supplier play an alternating-offer game with exogenous probability of breakdown as in Binmore, Rubinstein and Wolinsky (1986). The bargaining protocol implies a unique subgame-perfect equilibrium price that is identical for all suppliers and solves the generalized Nash bargaining problem over the match surplus.<sup>2</sup> This solution depends on the firm's infra-marginal productivities of capital because the firm realizes that if negotiations with any supplier break down, the price with all other matched suppliers is renegotiated so as to reflect the change in the firm's productivity. Furthermore, since capital is a durable factor of production, the effect of a larger capital stock today affects the firm's bargaining position with capital suppliers in future periods. As long as capital goods are substitutes, the firm thus has an incentive to overinvest relative to the socially optimal level because doing so drives down the productivity of new capital and, in turn, the negotiated price.

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<sup>1</sup>The formalization of trading frictions with random matching is closely related to the now widely used search and a matching approach for the labor market (e.g. Mortensen and Pissarides, 1994), and is similar in spirit to the modelization of trading frictions for financial assets (e.g. Wasmer and Weil, 2004; Duffie, Garleanu and Pedersen, 2005; Lagos and Rocheteau, 2008b).

<sup>2</sup>As Stole and Zwiebel (1996a) show, the non-cooperative solution implied by their pairwise bargaining protocol is equivalent to the Shapley values of a corresponding cooperative game. This provides an appealing economic reinterpretation of the bargaining protocol.

The random matching model to formalize ex-post specificity is appealing because it implies a tractable analytical solution and represents a minimal departure from the neoclassical benchmark. Furthermore, Kurmann and Petrosky-Nadeau (2007) and Gavazza (2008) show that random matching captures different empirical regularities about the allocation and reallocation of physical capital. At the same time, the model embodies a number of important assumptions and it is legitimate to ask to what extent the overinvestment result is robust to alternative environments. Section 4 addresses this concern. First, I show that the overinvestment result emerges independently of the matching friction as long as one maintains that specificity prevents firms from costlessly replacing a capital supplier with another. At the same time, matching frictions play an important role in general equilibrium because firms and suppliers do not take into account the effect of their investment decision on match probabilities. Together with the holdup problem, this externality implies that the decentralized allocation is always inefficient; and, depending on the severity of the externality, may even cause investment in general equilibrium to fall below the efficient level. This underscores the importance of working with a microfounded model of specificity when assessing the welfare implications of holdup problems.

Second, I alter the characteristics of the capital suppliers. Following the neoclassical benchmark, capital suppliers are modeled as atomistic agents who each produce a homogenous, perfectly divisible capital good prior to entering the market. In reality, capital suppliers are often large entities who produce heterogenous capital goods and negotiate with firms simultaneously over price and quantity. I show that the overinvestment result survives in such an alternative environment as long as one maintains sequentiality of the firm's investment decisions and substitutability across capital goods. Sequentiality ensures that the firm cannot enter into ex-ante contracts with future suppliers. Substitutability, in turn, implies that investment in a given capital good lowers the marginal productivity of other capital goods. As in the basic model, the firm therefore has a strategic motive to overinvest in capital because a larger capital stock today improves the firm's bargaining position with future suppliers.

Section 5 assesses how holdup problems in physical capital markets interact with specificity and contracting problems in the labor market. To this end, I incorporate the capital matching environment in a general equilibrium macroeconomic framework with labor market matching frictions and ex-post bargaining along the lines of Mortensen and Pissarides (1994). The resulting decentralized

economy is characterized by a double holdup problem. As before, holdups in physical capital markets provide the firm with an incentive to overinvest. But now, firms also face a holdup problem in the labor market that provides them with an incentive to overhire so as to lower labor productivity and thus the negotiated wage. This overhiring result is exactly the case emphasized by Stole and Zwiebel (1996a,b) and Smith (1999), and extended more recently by Cahuc, Marque and Wasmer (2007). Under the assumption that capital and labor are substitutes in production, the two holdup problems lead to both overinvestment and overhiring, thus offering a cautionary tale that holdup problems in the labor market do not necessarily lead to underinvestment, as is typically emphasized in the literature (e.g. Grout, 1984; Acemoglu and Shimer, 1999). If, instead, capital and labor are complements, the two holdup problems partly neutralize each other and present the firm with a trade-off. Overinvestment reduces the holdup problem in the capital market but exacerbates the holdup problem in the labor market. Vice versa, overhiring alleviates the holdup problem on the labor side but worsens the holdup problem on the capital side. Which side of this trade-off prevails depends on the specifics of technology and the relative bargaining powers of capital suppliers and workers. The two holdup problems thus neutralize each other at least partially and imply that the decentralized economy may be relatively close to efficiency even if each holdup problem on its own severely distorts allocations.

Section 6 concludes by discussing the importance of the overinvestment result for optimal policy, and why overinvestment may offer interesting explanations for a number of empirical observations.

## **2 Related literature**

The overinvestment result contrasts with much of the existing literature on holdup problems even though the underlying causes – factor specificity and incomplete contracts – are the same. The difference in results is due to small but important alternative assumptions about the microstructure behind specificity and contract incompleteness. In classic studies on holdup problems, it is assumed that while agents can negotiate in competitive markets prior to making investment decisions, institutions are insufficient to prevent ex-post renegotiation. In response, a large literature has emerged that analyzes under what conditions alternative arrangements help circumvent holdup

problems even in the presence of incomplete contracts.<sup>3</sup> In the present environment, the sequential nature of investment decisions prevents firms and capital suppliers from entering into contact with each other at the time the firm needs to sink resources. Hence, there is simply no counterparty with which to conclude a contract or any other related arrangement. This sequential assumption of 'invest-and-meet' is similar to descriptions of the labor markets where firms first have to invest before hiring workers, and workers first need to conclude their education before finding jobs.<sup>4</sup> As Acemoglu and Shimer (1998) show, the only way in which holdup problems are prevented in such a situation is if either party can commit ex-ante to a price and the other party can direct their search towards the agent that offers the best trade-off between price and match probability. As I show in Section 4, the same price posting / directed search assumption resolves the holdup problem in the present context and results in an efficient allocation. It remains an open question, however, what type of institutions or mechanisms prevent firms from renegeing ex-post on their posted prices – especially in a world with highly specialized capital projects that need to be frequently adjusted to changing situations.<sup>5</sup>

The overinvestment result also contrasts with Caballero and Hammour (1998a) who analyze an environment that looks very similar in the sense that the value of factors is higher in joint production than in autarky. Yet, Caballero and Hammour find that contract incompleteness in their model leads to underinvestment. Two assumptions explain this difference. First, Caballero and Hammour impose that the productivity in joint relationships is exogenous and thus, the concept of the firm as an independent optimizing entity is absent. In other words, their model assumes away any role for decreasing marginal returns (or more generally, substitutability across capital goods) to affect the bargaining set and thus the negotiated price of capital. Second, Caballero and Hammour

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<sup>3</sup>Prominent examples include the ex-ante reallocation of property rights and vertical integration (e.g. Williamson, 1975; Klein, Crawford and Alchian, 1978; Grossman and Hart, 1985; Hart and Moore, 1990); punishment schemes (MacLeod and Malcolmson, 1993); or long-term relations (Williamson, 1975). See Holmstrom and Roberts (1998) and Caballero (2007) for reviews.

<sup>4</sup>The assumption of investment in capital goods and projects prior to trading is also similar in spirit to the analysis of innovation and venture capital cycles by Silveira and Wright (2005, 2006).

<sup>5</sup>More generally, Caballero (2007, page 64) argues that *"...specific investments are typically made not once and for all, but incrementally throughout the life of a production unit. [In ideal contracts], the plan for making such investments, the duration of the relationship, the rent-division mechanism, and the multiple dimensions that characterize each factor's participation, must be pre-specified from the start and made fully contingent on the future profitability of the production unit, on factors that determine its evolving prospects, and on the various events, both aggregate and idiosyncratic, that govern each factor's outside opportunity costs. A variety of problems of observability, verifiability, enforceability, and sheer complexity, make such ideal contracts rarely feasible. Thus, agents enter into arrangements...that leave plenty of room for ex post discretion."*

assume that factor suppliers have a convex cost structure and "...that each type of factor in the production unit forms a coalition that bargains as a single agent" (fn. 13, p. 733). Their analysis thus puts all the power to alleviate the holdup problem in the hands of the suppliers. In the present framework, by contrast, firms exploit substitutability across capital goods to attenuate the holdup problem. Suppliers, in turn, operate in a decentralized market where they match with firms that carry over a large part of their capital stock from previous periods. Perfect coalitions among suppliers in such a setting seems all but impossible since this would imply that all suppliers over the entire history of a firm's capital accumulation collaborate ex-ante (i.e. before they even know each other) over the bargaining strategy.<sup>6</sup> One interpretation of this difference in assumptions is that Caballero and Hammour's environment is one about holdup problems for firm entry whereas the present environment is about holdup problems in the allocation of capital to existing firms (which, empirically, hold the majority of physical capital). This comparison highlights the crucial role of microeconomic structure for the analysis of holdup problems, and one of the main contribution of the paper is to show that two relatively innocuous assumptions – substitutability in production and absence of perfect coalitions among capital suppliers – overturn Caballero and Hammour's underinvestment result.

Finally, the overinvestment result contrasts with studies by Acemoglu and Shimer (1999) or Aruoba, Waller and Wright (2006) where – although for very different reasons – holdup problems imply underinvestment despite the fact that suppliers do not collude and capital is subject to diminishing returns. The main reason for this difference is that capital allocation in these studies is frictionless.<sup>7</sup> Instead, holdup problems occur because firms need to hire labor (respectively sell goods) in frictional markets that are characterized by incomplete contracts. Firms in such a situation want to underinvest because this lowers labor productivity (respectively increases the marginal cost of production), thus affecting the bargaining set over which they negotiate ex-post the wage (the price) with the worker (the consumer). The point of the paper is not to question the mechanism behind underinvestment that these studies emphasize. Rather, the objective of the paper is to analyze how trading frictions in physical capital markets by themselves can lead to

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<sup>6</sup>As I show in Section 4, coalition of suppliers *within* a given period does not negate the overinvestment result because firm's still have a strategic motive for investment *across* periods.

<sup>7</sup>More precisely, capital in these papers is specific only in the sense that investment needs to take place prior to production. The price of capital, however, is determined competitively and thus, there is no holdup problem in the market for physical capital.



overinvestment. In this sense, the paper is closely related to the analysis of intra-firm bargaining in labor markets by Stole and Zwiebel (1996a,b). Holdup problems in their environment arise because of specificity in firm-worker relationships and the absence of ex-ante binding contracts.<sup>8</sup> Multi-worker firms can react strategically to this holdup problem by increasing employment as this drives down the marginal productivity of labor and thus the bargained wage rate. Without hiring costs, Stole and Zwiebel (1996a) show that firms overhire to drive wages all the way down to the workers' reservation wage. Smith (1999) and more recently Cahuc, Marque and Wasmer (2007) incorporate this mechanism into a modern matching framework and show that in this case, hiring costs prevent firms from completely eliminate the holdup problem. Furthermore, these papers show that Stole and Zwiebel's overhiring result does not necessarily survive in general equilibrium because of the additional distortions associated with matching. In the present paper, the same result obtains: externalities related to trading frictions may outweigh the overinvestment incentive from the holdup problem and lead to inefficiently low capital accumulation.

### 3 Overinvestment in a basic model

The basic model is intentionally kept simple to convey the main intuition. Physical capital is the only factor of production and the opportunity cost of capital supply is exogenous. A more general model with labor as a second factor of production and an endogenous consumption-capital supply margin is analyzed in Section 5.

#### 3.1 Environment

The basic model is populated by a continuum of capital suppliers and a continuum of firms. All agents live forever in discrete time and do not discount the future. There is some general consumption good from which both capital suppliers and firms derive linear utility. Capital suppliers are atomistic. Each one of them can transform consumption goods into new capital goods by means

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<sup>8</sup>Note that there is a subtle but important difference in contract incompleteness between the present case and the one in Stole and Zwiebel (1996a,b), Smith (1999) and Cahuc et al. (2007). In their papers, overhiring is derived under the assumption that all labor contracts are renegotiated every period and that there is no mechanism to commit to future wage and employment decisions (i.e. employment relationships are 'at-will'). In the present paper, capital is purchased and thus, renegotiation in the future is not an issue. Instead, contract incompleteness arises naturally because of the sequentiality of investment decisions.

of a linear technology. Firms produce consumption goods with productive capital stock  $k$  using strictly increasing and concave technology  $f(k)$  that satisfies the usual Inada conditions.

The allocation of physical capital from suppliers to firms is subject to matching frictions along the lines of Kurmann and Petrosky-Nadeau (2007) and occurs in two phases. In the first phase, firms open 'ventures' at flow cost  $\gamma$  and search for new capital. Capital suppliers, in turn, enter the market with new (i.e. 'liquid') capital and search for firms with open ventures. Let  $\bar{v}$  denote the total number of new ventures by all firms in the current period, and  $\bar{l}$  the total number of liquid units of capital available. Then, if suppliers do not direct their capital towards any particular (group of) firms,  $\theta = \frac{\bar{v}}{\bar{l}}$  describes the capital market tightness, and total capital additions are governed by some matching process  $m(\bar{v}, \bar{l}) \leq \min(\bar{v}, \bar{l})$ , with  $\lim_{\theta \rightarrow 0} \partial m(\bar{v}, \bar{l}) / \partial \bar{v} = 1$  and  $\lim_{\theta \rightarrow \infty} \partial m(\bar{v}, \bar{l}) / \partial \bar{v} = 0$ . Accordingly, each venture matches with a unit of new capital with probability  $p(\theta) = m(\bar{v}, \bar{l}) / \bar{v}$ , and each unit of new capital matches with a venture with probability  $q(\theta) = m(\bar{v}, \bar{l}) / \bar{l}$ . Firms and capital suppliers take capital market tightness and thus the matching probabilities as exogenous. Following most of the random matching literature,  $m(\cdot, \cdot)$  is assumed constant returns to scale and thus,  $p(\theta)\theta = q(\theta)$ . For the subsequent analysis, it will also be useful to define the elasticity of matching with respect to liquid capital as  $\frac{\partial m(\bar{v}, \bar{l})}{\partial \bar{l}} \frac{\bar{l}}{m(\bar{v}, \bar{l})} = \frac{m_l}{q(\theta)} = \epsilon(\theta)$ , and the elasticity of matching with respect to projects as  $\frac{\partial m(\bar{v}, \bar{l})}{\partial \bar{v}} \frac{\bar{v}}{m(\bar{v}, \bar{l})} = \frac{m_v}{p(\theta)} = 1 - \epsilon(\theta)$ .<sup>9</sup>

In the second phase, ventures with matched capital become productive and, together with the existing capital stock, yield output. After production has taken place, some exogenous fraction  $\delta$  of the firm's capital stock disappears due to depreciation. Unmatched capital, in turn, remains idle for the period and then returns to the capital suppliers for consumption, net of some fraction  $(1 - \zeta)$ .<sup>10</sup>

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<sup>9</sup>To provide a specific example of matching in physical capital market that exhibits constant-returns-to-scale and satisfies the boundedness conditions, consider a formulation that has been used previously by Den Haan, Ramey and Watson (2000) for the labor market. For the present context, this matching function takes the form

$$m(\bar{v}, \bar{l}) = \frac{\bar{v}\bar{l}}{(\bar{l}\chi + \bar{v}\chi)^{1/\chi}}$$

with  $\chi > 1$ . The denominator  $J \equiv (\bar{l}\chi + \bar{v}\chi)^{1/\chi}$  denotes the number of submarkets in which the capital market is segmented. Available capital  $\bar{l}$  and new ventures  $\bar{v}$  are assigned randomly to one of the submarkets. Once assigned, technological and spatial constraints prevent  $\bar{l}$  and  $\bar{v}$  from moving to another submarket. These constraints are the source of market segmentation and, together with the random assignment assumption (due, for example, to information imperfections about potential suppliers and firms), give rise to the matching friction. A match occurs when a capital supplier and a venture are in the same submarket. The other ventures and capital suppliers remain unmatched. Under these assumptions, the probability of a capital supplier to match with a venture is  $\bar{l}/J$ , the probability of a venture to match with available capital is  $\bar{v}/J$ , and the total number of matches is  $\bar{v}\bar{l}/J$ .

<sup>10</sup>The presence of this deadweight loss is not essential for the results, except that in the present simple model with

Given these assumptions, the evolution of the productive capital stock of a firm is described by

$$k_{+1} = (1 - \delta)k + p(\theta)v, \quad (1)$$

where  $v$  denotes the number of new ventures per firm. Likewise, the evolution of idle capital is

$$u_{+1} = (1 - q(\theta))l, \quad (2)$$

where  $l$  denotes the number of capital units available for matching per firm; and  $u$  denotes the number of idle capital units per firm.

### 3.2 Efficient allocation

An allocation in this environment is (constrained) efficient if it maximizes the discounted sum of total consumption goods subject to the technological constraint and the capital allocation friction. Consider thus a hypothetical social planner who solves the following problem

$$O(k, u) = \max_{l, v} [f(k) + \zeta u - \gamma v - l + O(k_{+1}, u_{+1})]$$

subject to

$$\begin{aligned} k_{+1} &= (1 - \delta)k + m(v, l) \\ u_{+1} &= l - m(v, l). \end{aligned}$$

Since the social planner takes into account externalities from the matching friction and all firms are identical, the problem is directly formulated in terms of aggregates  $\bar{l} = l$ ,  $\bar{v} = v$ ,  $\bar{k} = k$ .<sup>11</sup>

The analysis focuses exclusively on steady state equilibria, where  $x = x_{+1}$  for all involved variables. Appendix A provides an explicit derivation of the social planner's problem. Here, I simply focus on the characterization of the resulting equilibrium and provide some intuition.

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linear utility and no discounting, it introduces an opportunity cost for the capital supplier such that the equilibrium is well-defined.

<sup>11</sup>I assume that firms are large enough such that the expected number of matches equals the realized number of matches. I thus abstract from any size and aggregation issues. In the full model in Section 5, aggregation and firm size are unimportant because production is constant returns to scale in capital and labor.

**Proposition 1.** *There exists a unique efficient equilibrium, characterized by the solution  $(k^S, \theta^S)$  to the following optimality conditions*

$$\gamma = m_v \left[ \frac{f'(k)}{\delta} - \zeta \right] \quad (3)$$

$$1 - \zeta = m_l \left[ \frac{f'(k)}{\delta} - \zeta \right] \quad (4)$$

PROOF: Appendix A.

Equation (3) says that the cost of an additional venture equals the marginal increase in matches with respect to ventures times the marginal net surplus of an additional match. Equation (4), in turn, says that the opportunity cost of converting a unit of consumption into additional liquid capital equals the marginal increase in matches with respect to liquid capital times the net surplus of an additional match. Combining the two equations to eliminate the marginal net surplus, and recognizing that  $m_v = (1 - \epsilon(\theta))p(\theta)$  and  $m_l = \epsilon(\theta)q(\theta) = \epsilon(\theta)p(\theta)\theta$ , yields the following solution for the socially efficient capital market tightness<sup>12</sup>

$$\theta^S = \frac{1 - \zeta}{\gamma} \frac{1 - \epsilon(\theta)}{\epsilon(\theta)} \in [0, \infty]. \quad (5)$$

Given  $\theta^S$ , either (3) or (4) then pins down the socially efficient capital stock  $k^S$ . This equilibrium is depicted by the dark blue lines in Figure 1.

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<sup>12</sup>As Arseneau, Chugh and Kurmann (2008) show, this condition describes the equivalence of intertemporal margins of transformation and substitution that, similar to the frictionless neoclassical benchmark, ensures efficiency.

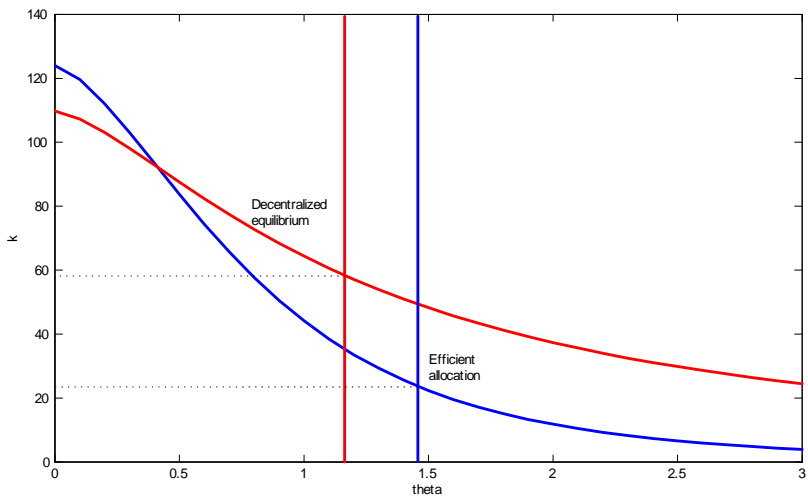


Figure 1: Efficient and decentralized equilibrium in the basic model

### 3.3 Decentralized equilibrium with ex-post bargaining

In the decentralized economy, firms purchase capital from their matched suppliers. By assumption of the matching friction, firms cannot contract on the price of capital ex-ante because they first need to open ventures before they enter the market to find capital suppliers. Following Stole and Zwiebel (1996a), I assume that once matched, suppliers and firms can reenter into pairwise renegotiation of the price at any time before production starts. Moreover, within each pairwise negotiation, the firm and the supplier play an alternating-offer game with exogenous probability of breakdown as in Binmore, Rubinstein and Wolinsky (1986). As Stole and Zwiebel show, this bargaining protocol implies a unique subgame-perfect equilibrium price that is identical for all suppliers, with the price solving the generalized Nash bargaining problem over the match surplus.<sup>13,14</sup> This solution depends on the firm's and the capital suppliers' relative bargaining power as well as the value of their respective outside options. For the capital suppliers, the outside option is to have an unmatched unit of capital that is returned to them in the following period net of loss  $1 - \zeta$ . For the firm, the

<sup>13</sup>Wolinsky (2000) confirms the existence and uniqueness of a stationary equilibrium in a dynamic context. Also, all of the following results would be robust to an alternative sharing rule over the gross surplus proposed by Shaked and Sutton (1984). See Stole and Zwiebel (1996a, Appendix B) for details.

<sup>14</sup>The assumption of pairwise renegotiation implies that suppliers do not cooperate with each other in the negotiation with the firm. While this assumption may seem strong, Stole and Zwiebel (1996a) demonstrate that their solution is equivalent to the Shapley values of a corresponding cooperative game. This equivalence is appealing as it implies Shapley values for any ordering of suppliers and not just the expectation over a randomized order.

outside option is to produce with one unit less of capital, which affects its marginal productivity. Hence, in the eyes of the firm, the rental rate is a function of its capital stock and thus expressed as  $\rho = \rho(k)$ .

Consider a capital supplier that enters the capital market by transforming a unit of consumption good into a unit of new capital good. The value of this entry decision is given by (ignoring time subscripts)

$$W_e = -1 + q(\theta)W_k + (1 - q(\theta))W_u, \quad (6)$$

where  $W_k$  and  $W_u$  denote the value of a matched and unmatched unit of capital next period, respectively. The former is defined by  $W_k = \rho$ , and the latter by  $W_u = \zeta$ . In optimum, there is free entry; i.e. capital suppliers find it optimal to provide a unit of new capital until  $W_e = 0$ .

Now, consider a firm that enters the period with capital stock  $k$ , of which  $k - (1 - \delta)k_{-1}$  is newly matched, and needs to decide on how many new ventures  $v$  to undertake. Its problem is described by

$$\begin{aligned} J(k, k_{-1}) &= \max_v [f(k) - \rho(k)[k - (1 - \delta)k_{-1}] - \gamma v + J(k_{+1}, k)] \\ \text{s.t. } k_{+1} &= (1 - \delta)k + p(\theta)v. \end{aligned}$$

The first-order condition is

$$\gamma = p(\theta) \frac{\partial J}{\partial k_{+1}}. \quad (7)$$

The marginal value of an additional unit of capital is given by the envelope condition

$$\frac{\partial J}{\partial k} = f'(k) - \rho'(k)[k - (1 - \delta)k_{-1}] - \rho(k) + \rho(k_{+1})(1 - \delta) + (1 - \delta) \frac{\partial J}{\partial k_{+1}}. \quad (8)$$

As discussed above, the firm takes into account that its outside option and thus the price of capital it negotiates with its new suppliers is affected by the capital stock. This strategic consideration is embodied in the term  $\rho'(k)[k - (1 - \delta)k_{-1}]$ .

The decentralized economy is closed with Stole and Zwiebel's (1996a) bargaining protocol. The renegotiation proof price that emerges is the solution to the generalized Nash bargaining problem

over the match surplus  $J_k + W_k - W_u$ . This price solves

$$\phi \frac{\partial J}{\partial k_{+1}} = (1 - \phi)(W_k - W_u), \quad (9)$$

where  $\phi$  denotes the bargaining power of the capital supplier.

To analyze the firm's strategic investment behavior, I combine equations (7) and (8) and apply the steady state condition  $k_{+1} = k = k_{-1}$  to obtain, after some rearrangement, an expression for the firm's demand for productive capital

$$\rho(k) = \frac{f'(k) - \rho'(k)\delta k}{\delta} - \frac{\gamma}{p(\theta)}. \quad (10)$$

Without the  $\rho'(k)\delta k$  term, this equation would simply be a break-even condition, saying that the price of capital equals the firm's present value of an additional unit of capital minus the average investment cost. With the  $\rho'(k)\delta k$  term, the decision becomes more involved. The firm realizes that adding capital affects the price for all new capital, now and in the future. To describe the impact of this addition, consider the Nash bargaining solution in (9). It implies the following expression for the price of capital (see Appendix A for details)

$$\rho(k) = \phi \left[ \frac{f'(k) - \rho'(k)\delta k}{\delta} \right] + (1 - \phi)\zeta.$$

The price is a weighted average of the maximum price  $[f'(k) - \rho'(k)\delta k]/\delta$  that the firm is willing to pay once matched, and the capital supplier's outside option  $\zeta$ , which equals the value of an unmatched unit of capital in case negotiations break down. This is a non-homogenous linear differential equation of  $\rho$  in  $k$  with solution

$$\rho(k) = k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1-\phi}{\phi}} \frac{f'(z)}{\delta} dz + (1 - \phi)\zeta. \quad (11)$$

The firm's maximum price is a weighted average of the marginal contribution of each unit of investment to the firm's present value. Intuitively, this dependence on infra-marginal productivities occurs because the firm realizes that in case of a negotiation breakdown with one of its capital suppliers, the firm's productivity changes from  $f'(k)$  to  $f'(k^-)$  and thus, the price with all other

suppliers can be renegotiated based on the new marginal productivity  $f'(k^-)$  – an argument that can be carried through all the way to the first unit of investment, in which case the firm’s capital stock is  $(1 - \delta)k$ . Of course, this consideration of infra-marginal productivities is relevant only because firms cannot costlessly replace a capital supplier with another in case negotiations break down. It is this assumption of capital specificity that provides the scope for strategic investment behavior.

Given the solution for the price of capital, the capital demand equation in (10) can be rewritten as

$$\rho(k) = OI \times \frac{f'(k)}{\delta} - \frac{\gamma}{p(\theta)}, \quad (12)$$

with  $OI \equiv 1 - \left[ k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1}{\phi}} f''(z) dz \right] / f'(k)$  denoting the ‘overinvestment factor’. If  $f''(\cdot) = 0$  for the range  $[(1 - \delta)k, k]$ , marginal productivity  $f'(\cdot)$  is unaffected by the firm’s investment behavior and thus, the overinvestment factor equals  $OI = 1$ . If marginal productivity is decreasing over some segment of  $[(1 - \delta)k, k]$ , however, then  $OI > 1$ . In this case, the firm finds it optimal to drive down productivity and thus the price of capital by increasing  $k$  beyond what is warranted by the productivity of the marginal capital unit. Proposition 2 sums up this result.

**Proposition 2.** *For  $f''(\cdot) \leq 0$  with strict inequality over some segment of  $[(1-\delta)k, k]$ , the firm overinvests relative to the social planner in order to reduce the price of capital.*

PROOF: Appendix A.

The overinvestment result can be nicely illustrated when technology takes the form  $f(k) = k^\alpha$ . Then, equation (12) becomes

$$\rho(k) = \frac{1 - \phi(1 - \alpha)(1 - \delta)^{\frac{1-\phi(1-\alpha)}{\phi}} \alpha k^{\alpha-1}}{1 - \phi(1 - \alpha)} \frac{\gamma}{\delta} - \frac{\gamma}{p(\theta)}.$$

The term  $\frac{\alpha k^{\alpha-1}}{\delta}$  denotes the firm’s present value of a marginal addition to its capital stock; and the overinvestment factor equals  $OI = [1 - \phi(1 - \alpha)(1 - \delta)^{\frac{1-\phi(1-\alpha)}{\phi}}] / [1 - \phi(1 - \alpha)]$ . For  $\alpha = 1$  (i.e.  $f''(\cdot) = 0$ ), marginal productivity is constant and there is no overinvestment. For  $\alpha < 1$ , the firm’s productivity depends on its capital stock and there is overinvestment as long as  $\phi > 0$ ; i.e.  $OI > 1$ . This overinvestment term increases with  $\phi$  as higher bargaining power for the capital



suppliers increases the firm's incentive to overinvest because the capital supplier obtains a larger part of the match surplus.

Proposition 2 is a partial equilibrium result for a given level of  $\theta$  and  $\rho(k)$ . To examine the effects of overinvestment in general equilibrium, I combine the firm's capital demand in (10) with the price of capital in (11) to obtain

$$\gamma = p(\theta)(1 - \phi) \left[ OI \times \frac{f'(k)}{\delta} - \zeta \right]. \quad (13)$$

This equation implies that in equilibrium, the firm's cost per venture equals its share of the match surplus times the probability of a match. Likewise, the capital supplier's condition (6) and the free-entry condition  $W_e = 0$  can be combined with the rental rate equation to obtain

$$1 - \zeta = q(\theta)\phi \left[ OI \times \frac{f'(k)}{\delta} - \zeta \right]. \quad (14)$$

This equation implies that in equilibrium, the capital supplier's opportunity cost of a unit of capital equals its share of the match surplus times the probability of a match. The two equations lead to the following characterization of the decentralized Nash bargaining equilibrium.

**Proposition 3.** *There exists a unique decentralized bargaining equilibrium defined by  $(k^B, \theta^B)$  that solves (13) and (14). This equilibrium is always inefficient. In particular,*

- if  $\phi = \epsilon(\theta^S)$ , then  $\theta^B = \theta^S$  and  $k^B > k^S$ ;
- if  $0 < \phi < \epsilon(\theta^S)$ , then  $\theta^B > \theta^S$  and either  $k^B > k^S$  or  $k^B < k^S$ ;
- if  $\epsilon(\theta^S) < \phi \leq 1$ , then  $\theta^B < \theta^S$  and either  $k^B > k^S$  or  $k^B < k^S$ .

PROOF: Appendix A.

To provide some intuition for this proposition, combine (13) and (14) to eliminate the bracketed expression. This yields a unique solution for capital market tightness

$$\theta^B = \frac{1 - \zeta}{\gamma} \frac{1 - \phi}{\phi} \in [0, \infty]. \quad (15)$$

Given  $\theta^B$ , either (13) or (14) then pins down the decentralized steady state capital stock  $k^B$ . The light red lines in the above Figure 1 illustrate this equilibrium.

The decentralized solution for  $\theta^B$  in (15) closely resembles the socially efficient solution in (5), with the difference that the bargaining power  $\phi$  replaces the matching elasticity  $\epsilon(\theta)$ . If  $\phi = \epsilon(\theta)$ , then  $p(\theta)(1 - \phi) = m_v$  and  $q(\theta)\phi = m_l$ , in which case the bargaining power is just strong enough that the firm and the capital supplier's matching probabilities reflect the externality from adding another venture, respectively another unit of liquid capital. This is the equivalent of Hosios' (1990) famous efficiency condition in search models of the labor market with bargaining. Comparing (13) and (14) with the social planner counterparts in (3) and (4), it is easy to see that the decentralized allocation is efficient if and only if  $OI = 1$ ; i.e. if  $f''(\cdot) = 0$  and thus, there is no overinvestment. Assuming instead that  $f''(\cdot) < 0$  over at least some segment  $[(1 - \delta)k, k]$  (an assumption that is necessary for  $k$  to be well-defined),  $OI > 1$  which implies that  $k^B > k^S$  as long as  $\phi$  is sufficiently close to  $\epsilon(\theta)$ . As  $\phi$  drops below  $\epsilon(\theta)$ , suppliers get increasingly discouraged from entering. Likewise, as  $\phi$  increases above  $\epsilon(\theta)$ , firms become increasingly discouraged from posting ventures. At some point, the negative effect on the capital stock overwhelms the positive effects from the holdup problem and equilibrium capital drops below the socially efficient level.

## 4 Behind overinvestment

The overinvestment result hinges on the absence of ex-ante binding contracts and the presence of match specificity. This section illustrates the role of these two determinants by working through different alternatives. First, I analyze the model's prediction in a world with complete ex-ante contracts. Second, I show that while explicit trading frictions have important general equilibrium implications, they are not crucial for the overinvestment result as long as one maintains the assumption of specificity. Third, I discuss the robustness of the overinvestment result with respect to a number of changes to production and the characteristics of capital suppliers.

### 4.1 The role of contract incompleteness

In the basic model, capital suppliers and firms cannot draw up ex-ante contracts because they do not know their trading partner at the time of investment. Here, I consider an alternative environment where, for unspecified reasons, firms are able to commit to price of capital *before* matching with their suppliers. Capital suppliers, in turn, observe the posted prices and direct their search towards

the firm with the most favorable trade-off between match probability and price. All the other details of the model remain the same. This price posting assumption is similar to the ones proposed in Shimer (1996), Moen (1997) and Acemoglu and Shimer (1999) for the labor market.<sup>15</sup>

To formalize the new environment, each firm is interpreted as a submarket for capital, denoted by  $j$ . Upon observing the different price postings  $\rho^j$ , capital suppliers decide in which submarket they want to send their unit of new capital. Firms that post a higher price will attract more capital suppliers; i.e. capital market tightness  $\theta^j = v^j/l^j$  in submarket  $j$  is decreasing in  $\rho^j$ , where  $v^j$  is the number of new ventures by firm  $j$  and  $l^j$  is the number of capital suppliers that decided to send their capital to that firm. Capital suppliers thus face a trade-off between a higher price if matched and a lower probability of matching  $q(\theta^j)$ . Vice versa, firms realize that a higher price posting will attract more capital suppliers, thus increasing their matching probability  $p(\theta^j)$ .

Since firms are identical, they all make the same optimal decisions in equilibrium. I thus omit subscripts in the interest of simplification and, as before, work directly with a representative firm. Consider first the capital supplier's situation. Its optimal entry decision is identical to the one described in (6), only that now, there is such a condition for each submarket  $j$ . Combining these equations with the free-entry condition and the values of matched and unmatched capital, I obtain

$$1 - \zeta = q(\theta)(\rho - \zeta). \quad (16)$$

This equation defines market tightness as a function of the price, i.e.  $\theta = \theta(\rho)$ , and describes how capital supply reacts to different prices. The firm's problem, in turn, consists of deciding simultaneously on the number of new ventures and the posted price that attracts the optimal amount of capital suppliers; i.e.

$$J(k, k_{-1}) = \max_{v, \rho} [f(k) - \rho[k - (1 - \delta)k_{-1}] - \gamma v + J(k_{+1}, k)]$$

subject to the capital accumulation constraint  $k_{+1} = (1 - \delta)k + p(\theta)v$ , and the optimal capital supply relation in (16). Note that in this formulation, the price  $\rho$  no longer depends on  $k$  in the eyes of the firm, because  $\rho$  is now a choice variable.

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<sup>15</sup>As Acemoglu and Shimer (1999) show, the directing agent does not need to observe all price postings. Instead, Bertrand competition ensures that all results obtain even if the directing agent observes only two random postings.

An equilibrium in this environment must satisfy the following conditions: (i) entering capital suppliers earn zero profits; (ii) capital suppliers direct search towards the firm with the highest expected payoff; (iii) firms are profit maximizing; and (iv) the equilibrium is consistent with rational expectation; i.e. prices and probabilities are such that capital suppliers are indifferent about where to direct capital. In other words, if a firm posted another price, the trade-off between queue length and price would be such that capital suppliers would not want to apply. Under these conditions, the following proposition obtains.

**Proposition 4.** *There exists a unique decentralized price posting equilibrium that coincides with the efficient allocation  $(\theta^S, k^S)$ .*

PROOF: Appendix B.

This proposition is reminiscent of Acemoglu and Shimer (1999) and illustrates that one of the root causes of inefficiency in the decentralized economy is the absence of non-binding ex-ante contracts. Instead, if firms can commit ex-ante to a posted price, then they internalize both the externality of the matching friction and the holdup problem. As discussed in the literature review, however, it remains an open question to what extent mechanisms and institutions in physical capital markets exist that prevent renegotiation of ex-ante prices.

## 4.2 The (un)importance of explicit trading frictions

The main assumption of the paper is that the specificity inherent in most physical capital gives rise to trading frictions that render physical capital markets decentralized. It is important to emphasize, however, that holdups and overinvestment in partial equilibrium emerge independently of the exact form of the trading friction. To illustrate this point, I consider the limiting case without random matching frictions; i.e. there are no venture costs (i.e.  $\gamma = 0$ ), firms thus open an infinity of projects (i.e.  $\theta = \infty$ ) and all liquid capital is matched (i.e.  $q(\theta) = 1$ ). As in Stole and Zwiebel (1996a,b), however, I maintain ex-post specificity; i.e. markets remain decentralized such that firms cannot replace one capital supplier with another in case negotiations break down. The following proposition summarizes the implications of this special case.

**Proposition 5.** *Given  $f''(\cdot) \leq 0$  with strict inequality over some segment of  $[(1-\delta)k, k]$  and positive*

bargaining power by workers (*i.e.*  $\phi > 0$ ), the firm overinvests ( $k^B > k^S$ ) and the decentralized equilibrium remains inefficient.

PROOF: Appendix B.

To shed more light on this proposition, note that for  $\gamma = 0$ , the social planner solution collapses to

$$1 = f'(k) + (1 - \delta). \quad (17)$$

Without matching frictions, efficient capital accumulation is governed by the familiar Euler condition of the neoclassical benchmark (given zero discounting) that equates the intertemporal marginal rate of transformation with the intertemporal marginal rate of substitution. The decentralized equilibrium, in turn, reduces to

$$\rho = 1 \quad (18)$$

$$1 = OI \times f'(k) + 1 - \delta. \quad (19)$$

Equation (18) is implied by the capital suppliers' free entry condition: since the matching is frictionless and there is no discounting, the price of capital equals its opportunity cost, which is one. Equation (19), in turn, is derived from the firm's capital demand.<sup>16</sup> Because of ex-post specificity, firms still have the incentive to overaccumulate capital so as to drive down the price of capital, *i.e.*  $OI > 1$ , and the economy remains inefficient. The main result of the paper is therefore robust to whether ex-post specificity is modeled explicitly (*e.g.* through a matching friction) or assumed implicitly as in Stole and Zwiebel. Only if one departs from the assumption of specificity altogether and assumes competitive capital markets instead do holdups disappear. In this case, the market for physical capital effectively becomes competitive because firms can immediately replace one supplier with another.

The explicit formulation of trading frictions as the source of specificity is useful, nevertheless. First and as discussed above, trading frictions turn out to be crucial for efficiency in general equi-

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<sup>16</sup>Note that for general  $\phi$ , this solution is consistent with the bargaining solution for the price only if  $\zeta = 1$ . This should not be surprising. When matching is frictionless and suppliers do not discount time, the reservation value of a unit of unmatched capital must necessarily be 1. For  $\zeta < 1$ , the bargaining solution becomes an inequality because the capital supply condition  $\rho = 1$  implies that the firm cannot accumulate sufficient capital to drive down the price all the way to  $\zeta$  (which is the partial equilibrium result highlighted in Stole and Zwiebel, 1996a).

librium. Indeed, externalities from matching may be so important that equilibrium investment falls below the efficient level. Second, an explicit formulation of trading frictions helps to clarify the effects of specificity. In the basic model, specificity comes in two different forms: as a venture cost  $\gamma > 0$  that prevents some of the capital suppliers from being matched; and as a deadweight loss  $(1 - \zeta) > 0$  in case a capital unit remains unmatched or negotiations break down. The following proposition summarizes the effects of these two sources of specificity on allocations.

**Proposition 6.** *Venture costs  $\gamma$  have a negative effect on steady state capital stocks, while deadweight losses  $(1 - \zeta)$  have an ambiguous effect on steady state capital stocks.*

PROOF: Appendix B.

An increase in  $\gamma$  implies a larger average posting cost per new unit of matched capital. The effect is the same than a larger investment adjustment cost in classical models: firms embark on fewer ventures, which results in a lower steady state capital stock. A larger deadweight loss  $(1 - \zeta)$ , on the one hand, lowers the outside option of the capital supplier and thus, the price of capital. As a result, firms post more new ventures, which has a positive effect on steady state capital stocks. On the other hand, a larger  $(1 - \zeta)$  reduces the incentives of capital suppliers to participate in the market, which has a negative effect on steady state capital stocks. The overall effect is thus ambiguous.

### 4.3 Robustness to alternative modeling assumptions

Aside from contract incompleteness and ex-post specificity, the basic model embodies a number of important assumptions about production and the characteristics of capital suppliers. These assumptions are made to keep the analysis tractable, and the model comparable with the neoclassical benchmark.<sup>17</sup> Yet, it is legitimate to ask to what extent the overinvestment result is robust to alternative, more realistic assumptions.

Consider production first. In the basic model, capital is a perfectly divisible and homogenous input to production. In reality, capital projects are often indivisible and heterogenous. Stole

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<sup>17</sup>Note that in the neoclassical benchmark, firms rent rather than purchase capital. It is straightforward to show that overinvestment and inefficiency in general equilibrium occur in much the same way when firms rent matched capital, and rental rates are renegotiated on a period-per-period basis. Analytical details are available upon request.

and Zwiebel (1996a) derive their overhiring result in a discretized environment and thus, factor indivisibilities per se are not a problem. Stole and Zwiebel also demonstrate that their results generalize elegantly to a setup with heterogenous inputs as long as the different inputs remain substitutes of each other and each supplier has the same degree of bargaining power. Heterogeneity becomes more difficult to analyze when the bargaining power across suppliers varies. Cahuc, Marque and Wasmer (2007) provide a generalized treatment of these issues in a labor search model. Their analysis shows that the overhiring result of the homogenous worker case can be misleading. In particular, if a firm can substitute a worker group with higher bargaining power for other worker groups with less bargaining power, the optimal strategy for the firm is to underemploy the latter groups. A similar result would naturally emerge here. In all of this, it remains true, however, that strategic bargaining on part of the firm leads to overinvestment in some capital goods as long as there is at least some substitutability in the production function.<sup>18</sup>

Second, consider the characteristics of capital suppliers. In the basic model, suppliers are atomistic agents that need to pre-produce their capital goods and, once they match with a firm, do not cooperate with other suppliers in the bargaining. In reality, capital suppliers are often large entities who negotiate as a single party with the firm before actually sinking resources to produce capital.<sup>19</sup> To analyze the robustness of the overinvestment result to such an alternative market structure, I consider a simple static model with the following characteristics: (i) firms produce with two types of imperfectly substitutable capital goods  $k_1$  and  $k_2$  using technology  $f(k_1, k_2)$  that is concave in both arguments; (ii) each of the capital goods is produced by an independent supplier at some convex cost  $c(k_i)$ ; (iii) the market for  $k_1$  is perfectly competitive with price  $\rho_1$  while the market for  $k_2$  is decentralized with the firm and the supplier simultaneously negotiating over  $k_2$  and  $\rho_2$  using generalized Nash bargaining; (iv) investment is sequential in the sense that the firm acquires  $k_1$  in a first stage before it enters the market for  $k_2$  in a second stage.

Assumptions (i) and (ii) embody the idea that capital goods are heterogenous and supplied by large entities. Assumption (iii) introduces switching costs for  $k_2$ ; i.e. due to some unspecified

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<sup>18</sup>If all different capital goods are complements of each other, then there is a trade-off between overinvestment (lowering the marginal productivity of a given capital good) and underinvestment (lowering the marginal productivity of the other complementary capital goods). The outcome of this trade-off depends on the relative bargaining powers of the different suppliers and the specifics of the production function. Section 5 analyzes such a situation in a general model where capital and labor are complements.

<sup>19</sup>As discussed in footnote 14, the absence of cooperation among suppliers is by itself not important because resulting bargaining solution is equivalent to the Shapley values of a corresponding cooperative game.

trading friction, firms cannot replace their supplier for  $k_2$  with another one and vice versa, a supplier of  $k_2$  cannot sell its goods to another firm. As opposed to the basic model, however, capital suppliers do not need to sink resources ex-ante but produce capital only at the time the price is determined. Finally, the sequentiality assumption in (iv) provides the source of holdups. If the firm was able to invest simultaneously in both  $k_1$  and  $k_2$ , there would be no sense in which rents from ex-ante investments are shared ex-post and thus, there would be no holdups. With sequentiality, by contrast, the firm's ex-ante investment in  $k_1$  in the first stage affects the rent in the second stage and thus the bargained values of  $k_2$  and  $\rho_2$ . This bargaining necessarily occurs ex-post because at the time of the investment in  $k_1$ , the firm does not know yet its trading partner for  $k_2$ .

The efficient allocation in this model is straightforward: the marginal productivity of each capital good equals its marginal cost of production; i.e.  $\partial f/\partial k_i = \partial c/\partial k_i$  for both  $i = 1, 2$ . Given the characteristics of  $f(\cdot, \cdot)$  and  $c(\cdot)$ , the two conditions define a unique efficient equilibrium  $(k_1^S, k_2^S)$ .

In the decentralized world, the firm's problem in the first stage is

$$\max_{k_1} [f(k_1, k_2) - \rho_1 k_1 - \rho_2 k_2]$$

and the demand of  $k_1$  is given by

$$\rho_1 = \frac{\partial f(k_1, k_2)}{\partial k_1} - \frac{\partial \rho_2}{\partial k_1} k_2 - \rho_2 \frac{\partial k_2}{\partial k_1}. \quad (20)$$

The price of  $k_1$ , in turn, is given by  $\rho_1 = \partial c/\partial k_1$  by assumption of perfect competition in this market. The two additional terms in the firm's capital demand describe the firm's strategic behavior with respect to the holdup problem in the second stage. Similar to the baseline model with atomistic suppliers, the firm realizes that accumulating more  $k_1$  in the first stage will impact the marginal productivity of  $k_2$  and thus the bargaining in the second stage. By assumption (iii), the Nash bargaining in the second stage takes the form  $\max_{k_2, \rho_2} J^{1-\phi} W_2^\phi$ , where  $\phi$  is the bargaining power of the second-stage capital supplier;  $J = [f(k_1, k_2) - \rho_1 k_1 - \rho_2 k_2]$  is the value of the firm; and  $W_2 = \rho_2 k_2 - c(k_2)$  is the value of the second-stage capital supplier. The solution to this problem



is<sup>20</sup>

$$\frac{\partial f(k_1, k_2)}{\partial k_2} = \frac{\partial c(k_2)}{\partial k_2} \quad (21)$$

$$\rho_2 = \phi \left[ \frac{f(k_1, k_2) - \rho_1 k_1}{k_2} \right] + (1 - \phi) \frac{c(k_2)}{k_2}. \quad (22)$$

As long as  $\partial^2 f / (\partial k_1 \partial k_2) \neq 0$  (i.e. some substitution between  $k_1$  and  $k_2$  is possible),  $k_2$  and  $\rho_2$  are both functions of  $k_1$  and thus, the firm has a strategic motive in the first investment stage. Its implications are described by the following proposition.

**Proposition 7.** *There is a unique decentralized equilibrium  $(k_1^B, k_2^B)$  that solves (20)-(22). As long as  $f_{k_1 k_2} \neq 0$ , this equilibrium is always inefficient; with  $k_1^B > k_1^S$  iff  $\partial^2 f / (\partial k_1 \partial k_2) < 0$ , and  $k_1^B < k_1^S$  iff  $\partial^2 f / (\partial k_1 \partial k_2) > 0$ .*

PROOF: Appendix B.

Despite the difference in assumptions, the overinvestment result here is very similar to the overinvestment result in the baseline model with atomistic suppliers. As long as  $k_1$  and  $k_2$  are substitutes (i.e.  $\partial^2 f / (\partial k_1 \partial k_2) < 0$ ), the firm finds it optimal to overinvest in  $k_1$  so as to reduce the marginal productivity of  $k_2$  and, in turn, the price of capital  $\rho_2$  in the second stage. The result is an inefficiently high  $k_1$  and thus overinvestment.

The model here is certainly very stylized and could be expanded along different dimensions.<sup>21</sup> The point is simply to show that neither the assumption of atomistic suppliers nor the assumption of ex-ante commitment of resources by suppliers is crucial for the overinvestment result. Instead, and to sum up the discussion of this section, the crucial requirements behind the overinvestment result of this paper are:

1. sequentiality of investment decisions, preventing binding ex-ante contracts;
2. specificity of capital giving rise to decentralized bargaining;

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<sup>20</sup>The same solution could have been obtained alternatively by first maximizing the joint surplus  $J + W_2$  with respect to  $k_2$  and then splitting the surplus by maximizing the Nash product with respect to  $\rho_2$ . Hence, the solution is privately efficient.

<sup>21</sup>Most importantly, the model could be set in a dynamic context where firms sequentially invest in a number of different capital goods, with the price of each one of them being negotiated in a decentralized market.

3. substitutability across different capital inputs.

None of these assumptions seem particularly stringent compared to the assumptions usually made in the holdup literature.

## 5 Overinvestment and holdups in the labor market

In this section, I assess how holdup problems in physical capital markets interact with specificity and contracting problems in the labor market that typically lead to underinvestment. To do so, I incorporate the capital matching environment into a neoclassical framework where the labor market is also subject to random matching. The labor market friction is modeled as in the standard setup of Mortensen and Pissarides (1994), with the exception that firms are large and can hire multiple workers. Hence, in the absence of binding ex-ante contracts, the same strategic considerations with respect to holdup problems apply in the labor market than in the capital market, the consequences of which have been analyzed previously by Stole and Zwiebel (1996a,b), Smith (1999) and Cahuc, Marque and Wasmer (2007).

### 5.1 Environment

The model is populated by a large number of identical, infinitely-lived firms and households. Firms produce output  $y$  with matched capital  $k$  and hired labor  $n$  using a constant-returns-to-scale technology  $f(k, n)$  that is concave in both arguments and satisfies the usual Inada conditions. Households are composed of a continuum of workers and capital suppliers who discount the future at rate  $\beta = (1 + r)^{-1}$ .<sup>22</sup> Workers are either employed or unemployed and have logarithmic preferences for some general consumption good  $c$ .<sup>23</sup> The wage rate when employed is  $w$  and the flow value from non-market activity by the unemployed is  $b$ . Capital suppliers do not consume. For tractability, I assume as in the basic model that these suppliers are atomistic entities that do not cooperate once they are matched with a firm. Also, the spatial and technological constraints behind frictional matching make it impossible for an unmatched capital supplier to move its capital unit to another

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<sup>22</sup>For simplicity, I abstract from a distinct sector for capital allocation. Dropping this assumption is straightforward but would unnecessarily complicate the model.

<sup>23</sup>As will become clear, none of the analytical results are affected by the degree of risk aversion.

matched supplier from the same household. Households, finally, pool all revenues from workers and capital suppliers and allocate resources across consumption and investment in new capital units. As is well known from the macro literature on indivisible labor (e.g. Rogerson, 1988; or Hansen, 1985), optimal risk sharing under separable consumption preferences implies that every worker of the household consumes the same amount, disregarding of its employment history. The household therefore acts as a state-contingent insurance mechanism for its workers, and the environment can be analyzed as a standard representative agents model.<sup>24</sup>

The capital matching friction is the same than in the basic model. The only difference here is a slight change in notation to accommodate the relevant labor market variables. In particular, let  $v_k$  be the number of ventures posted per firm, and  $\gamma_k$  the flow cost per posting. Then,  $m(\bar{v}_k, \bar{l})$  describes the random matching process as a function of aggregate ventures  $\bar{v}_k$  and aggregate liquid capital  $\bar{l}$ . In turn,  $p(\theta_k) = m(\bar{v}_k, \bar{l})/\bar{v}_k$  and  $q(\theta_k) = m(\bar{v}_k, \bar{l})/\bar{l}$  denote the respective matching probabilities, with  $\theta_k = \bar{v}_k/\bar{l}$  measuring the capital market tightness. Matched capital joins the firm's existing capital stock for production in the next period. After production, a fraction  $\delta$  is lost to depreciation. In addition, a fraction  $s_k$  of the remaining capital stock separates thereafter and is sold back to the household at price  $0 \leq \varphi \leq 1$  per unit. The deadweight loss  $1 - \varphi$  is another source of specificity in the model.<sup>25</sup> Unmatched capital units, in turn, remain idle and return on the matching market in the next period, after incurring the same deadweight loss  $(1 - \zeta)$  than in the basic model. Given these assumptions, the representative firm's capital stock  $k$  evolves as

$$k_{+1} = (1 - s_k)(1 - \delta)k + p(\theta_k)v_k. \quad (23)$$

Likewise, unmatched capital per firm evolves as

$$u_{k,+1} = (1 - q(\theta_k))l. \quad (24)$$

On the labor market side, firms need to open job vacancies  $v_n$  at cost  $\gamma_n$  to search for available

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<sup>24</sup>Most of the general equilibrium labor literature adopts the same approach. See Andolfatto (1996), Alexopoulos (2004), or Gertler and Trigari (2008) for a few of many examples.

<sup>25</sup>The resale price  $\varphi$  is assumed exogenous here. One could easily endogenize this price to reflect that capital suppliers are willing to pay a higher price when the rematching probability is high. All the results in this sections would remain valid, with the exception that endogenizing  $\varphi$  affects the matching externality and thus Hosios' condition.

workers. Available workers are made up of unemployed workers from the previous period plus newly separated workers.<sup>26</sup> Let  $1 - \bar{n}$  denote the number of unemployed workers in the beginning of the period (with total workers normalized to one), and  $s_n \bar{n}$  the number of newly separated workers. Then  $1 - \bar{n} + s_n \bar{n}$  defines the number of available workers, and matching occurs according to some constant-returns-to-scale function  $m(\bar{v}_n, 1 - \bar{n} + s_n \bar{n})$ , with  $\bar{v}_n$  denoting aggregate job vacancies. Furthermore, define  $\theta_n = \bar{v}_n / (1 - \bar{n} + s_n \bar{n})$  as labor market tightness. Then  $p(\theta_n) = m(\bar{v}_n, 1 - \bar{n} + s_n \bar{n}) / \bar{v}_n$  and  $q(\theta_n) = m(\bar{v}_n, 1 - \bar{n} + s_n \bar{n}) / (1 - \bar{n} + s_n \bar{n})$  denote the probabilities of filling and finding a vacant job, respectively.<sup>27</sup> Once matched, workers enter production the next period. After production has taken place, workers separate at rate  $s_n$  and return, as just described, immediately into the labor market for rematching. Given these assumptions, the representative firm's employment  $n$  evolves according to

$$n_{+1} = (1 - s_n)n + p(\theta_n)v_n. \quad (25)$$

## 5.2 Efficient allocation

The constrained-efficient allocation for the representative agent solves the Bellman equation

$$O(k, u_k, n) = \max_{c, l, k, v_k, v_n} [\log c + \beta O(k_{+1}, u_{k,+1}, n_{+1})]$$

subject to the economy's resource constraint  $f(k, n) + b(1 - n) + (1 - \delta)s_k \varphi k + \zeta u_k \geq \gamma_k v_k + \gamma_n v_n + c + l$ ; the capital accumulation constraint  $k_{+1} = (1 - \delta)(1 - s_k)k + m(v_k, l)$ ; the evolution of unmatched capital  $u_{k,+1} = l - m(v_k, l)$ ; and the evolution of employment  $n_{+1} = (1 - s_n)n + m(v_n, 1 - n + s_n n)$ . Since the social planner takes into account the matching externality, the optimization problem is formulated directly over aggregates  $\bar{l} = l$ ,  $\bar{v}_k = v_k$ ,  $\bar{k} = k$ ,  $\bar{v}_n = v_n$ ,  $\bar{n} = n$ . As in the basic model, the analysis focuses exclusively on steady states where  $x = x_{+1}$  for all involved variables. The following proposition characterizes the efficient equilibrium allocation.

<sup>26</sup>The assumption that separated workers can reenter the labor market in the same period contrasts with the standard Mortensen-Pissarides setup, where separated workers reenter only the following period and thus spend at least one period in unemployment. I make this assumption here such that the model under zero matching frictions implies zero unemployment. Also see Den Haan, Ramey and Watson (2000) who make the same assumption.

<sup>27</sup>To avoid excessive notation, matching in both the capital and the labor market is described by  $m(\cdot, \cdot)$ , even though the functional form or parametrization is not necessarily the same. Hence, the probability measures  $p(\cdot)$  and  $q(\cdot)$  may differ between the two markets.

**Proposition 8.** *There is a unique efficient equilibrium, characterized by the solution  $(k^S, \theta_k^S, n^S, \theta_n^S)$  to the following conditions*

$$\gamma_k = m_{v_k} \beta \left[ \frac{\partial f(k, n) / \partial k + (1 - \delta) s_k \varphi}{1 - (1 - \delta)(1 - s_k) \beta} - \zeta \right] \quad (26)$$

$$1 - \beta \zeta = m_l \beta \left[ \frac{\partial f(k, n) / \partial k + (1 - \delta) s_k \varphi}{1 - (1 - \delta)(1 - s_k) \beta} - \zeta \right] \quad (27)$$

$$\gamma_n = m_{v_n} \beta \left[ \frac{\partial f(k, n) / \partial n - b}{1 - [1 - s_n - m_{u_n}(1 - s_n)] \beta} \right] \quad (28)$$

$$n = \frac{q(\theta_n)}{s_n + (1 - s_n)q(\theta_n)} \quad (29)$$

PROOF: Appendix B.

The first two equations are equivalent to the efficiency conditions for ventures and liquid capital in the basic model; i.e. conditions (3) and (4). Together, they pin down the capital market equilibrium, which is now defined by efficient capital market tightness  $\theta_k^S$  and an efficient capital labor ratio  $(k/n)^S$ . The third equation says that the cost of an additional job vacancy equals the marginal increase in labor market matches times the discounted surplus from an additional match. Together with the Beveridge curve in (29), which is simply the steady state version of the evolution of employment (25), this determines the efficient labor market tightness  $\theta_n^S$  and, in turn, the efficient levels of employment  $n^S$  and capital  $k^S$ .

### 5.3 Decentralized equilibrium with ex-post bargaining

As in the basic model, the decentralized economy is characterized by a complete absence of contract enforcement. At any time before production starts, firms can renegotiate the price of capital with their suppliers. Likewise, as in Stole and Zwiebel (1996a,b), wages of all workers (incumbents and newly matched) are renegotiated in the beginning of every period. Since both capital and labor are specific (due to the matching friction), this creates a double holdup problem. Similar to the basic model, the assumed bargaining protocol implies a renegotiation-proof price of capital and a renegotiation-proof wage rate that solves the generalized Nash bargaining problem over the respective surplus. These solutions depend on the outside options of the different parties. Since capital suppliers and workers are atomistic and do not collaborate in the ex-post bargaining, they

consider the price of capital, respectively the wage rate as exogenous. The firm, on the other hand, realizes that its outside option is to produce with one less unit of capital or labor, respectively. In the eyes of the firm, the price of capital and the wage rate therefore depend on capital and labor productivity, which are both functions of  $k$  and  $n$ .

The representative household enters the period with  $i$  newly matched units of capital,  $u_k$  unmatched units, and  $n$  employed workers. All of the  $1 - n$  unemployed and  $s_n n$  freshly separated workers participate in the labor market. The household thus has to decide on optimal consumption  $c$  and the mass of new suppliers it wants to send out into the capital market with liquid capital  $l$ . The Bellman equation of this problem is

$$\begin{aligned}
V(i, u_k, n) &= \max_{c, l} [\log c + \beta V(i_{+1}, u_{k,+1}, n_{+1})] \\
&\quad + \lambda [wn + b(1 - n) + \rho i + \zeta u_k + d - c - l] \\
\text{s.t. } i_{+1} &= q(\theta_k)l \\
u_{k,+1} &= (1 - q(\theta_k))l \\
n_{+1} &= (1 - s_n)n + \theta_n p(\theta_n)(1 - n + s_n n)
\end{aligned}$$

where  $\lambda$  is the Lagrangian multiplier on the household's budget constraint; and  $d$  are dividends from a perfectly diversified portfolio of firm ownership. The solution to this problem implies, after some rearrangement, the following steady state conditions (see Appendix B for details on all derivations in this section)

$$1 - \beta\zeta = q(\theta_k)\beta[\rho - \zeta]. \quad (30)$$

$$\frac{\partial V}{\partial n} = \frac{\lambda[w - R]}{1 - (1 - s_n)\beta}. \quad (31)$$

Except for the discount factor  $\beta$ , condition (30) is identical to (6) and the free-entry condition  $W_e = 0$  of the basic model. Condition (31), in turn, says that marginal value of employment equals the annuity value of wages net of the worker's reservation wage  $R \equiv b + (1 - s_n)q(\theta_n)\beta\frac{\partial V/\partial n}{\lambda}$ , which is defined as the flow benefits from unemployment,  $b$ , and the expected net value from employment in the next period.

On the firm's side, the optimal problem remains very similar to the basic model, only that now,

the firm also chooses new job vacancies  $v_n$ . The Bellman equation describing this problem is

$$\begin{aligned}
J(k, k_{-1}, n) &= \max_{v_k, v_n} \left[ \begin{aligned} &f(k, n) - \rho(k, n)[k - (1 - \delta)(1 - s_k)k_{-1}] + s_k(1 - \delta)\varphi k \\ &-w(k, n)n - \gamma_k v_k - \gamma_n v_n + \beta J(k_{+1}, k, n_{+1}) \end{aligned} \right] \\
\text{s.t. } k_{+1} &= (1 - \delta)(1 - s_k)k + p(\theta_k)v_k \\
n_{+1} &= (1 - s_n)n + p(\theta_n)v_n.
\end{aligned}$$

The notation  $\rho = \rho(k, n)$  and  $w = w(k, n)$  captures the fact that, as discussed above, the firm considers both the price of capital and the wage rate as functions of its capital stock and employment. The solution to the firm's problem yields, after some rearrangement, the following steady state expressions for capital demand and labor demand, respectively,

$$\rho(k, n) = \frac{\frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k}[1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial k}n + (1 - \delta)s_k\varphi}{1 - (1 - \delta)(1 - s_k)\beta} - \frac{\gamma_k}{\beta p(\theta_k)} \quad (32)$$

$$w(k, n) = \frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n}[1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial n}n - \frac{\gamma_n(r + s_n)}{p(\theta_n)}. \quad (33)$$

The terms  $\partial \rho / \partial k$  and  $\partial w / \partial k$  in (32) embody the firm's strategic motive with respect to capital accumulation; i.e. the firm realizes that an additional unit of capital affects negotiations with all newly matched suppliers as well as with all workers. The terms  $\partial \rho / \partial n$  and  $\partial w / \partial n$  in (33) capture the same strategic motive with respect to hiring. This makes clear that the holdup problem in the capital market not only has an effect on the firm's investment decision, but also contaminates the firm's hiring margin. Vice versa, the holdup problem in the labor market affects both the hiring and the investment decision of the firm.

The model is closed with the renegotiation-proof capital price and wage rate that, as discussed, emerge from the generalized Nash bargaining problem over the respective match surplus. The capital price solves  $\phi_k \frac{\partial J}{\partial k_{+1}} = (1 - \phi_k) \left[ \frac{\partial V}{\partial i} - \frac{\partial V}{\partial u_k} \right] / \lambda$  and the wage rate solves  $\phi_n \frac{\partial J}{\partial n} = (1 - \phi_n) \frac{\partial V}{\partial n} / \lambda$ , where  $\phi_k$  and  $\phi_n$  denote the bargaining power of the capital supplier and the worker, respectively. After some rearrangement, this yields equations for the price of capital and the wage rate that are

convex combinations of the respective parties' outside option

$$\rho(k, n) = \phi_k \left[ \frac{\frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial k} n + (1 - \delta)s_k \varphi}{1 - (1 - \delta)(1 - s_k)\beta} \right] + (1 - \phi_k)\zeta. \quad (34)$$

$$w(k, n) = \phi_n \left[ \frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial n} n \right] + (1 - \phi_n)R. \quad (35)$$

The two equations form a system of non-homogenous linear differential equations of  $\rho$  and  $w$  in  $k$  and  $n$ . Solving this system is non-trivial, but it turns out that one can adapt the spherical coordinate techniques used by Cahuc, Marque and Wasmer (2007) to derive a unique solution. In particular, define  $\tilde{\phi}_k = \phi_k/[1 - (1 - \delta)(1 - s_k)\beta]$ ,  $\tilde{\delta} = 1 - (1 - \delta)(1 - s_k)$  and  $\chi_{kn} = \frac{1 - \tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k} \frac{\phi_n}{1 - \phi_n}$ ,  $\chi_{nk} = 1/\chi_{kn} = \frac{1 - \phi_n}{\phi_n} \frac{\tilde{\delta}\tilde{\phi}_k}{1 - \tilde{\delta}\tilde{\phi}_k}$ . Then, the solutions for  $\rho(k, n)$  and  $w(k, n)$  are

$$\begin{aligned} \rho(k, n) &= \int_{\tilde{\delta}}^1 z^{\frac{1 - \tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k}} \frac{f_1(kz, nz^{\chi_{kn}})}{\tilde{\delta}} dz + \tilde{\phi}_k(1 - \delta)s_k\varphi + (1 - \phi_k)\zeta \\ w(k, n) &= \int_0^1 z^{\frac{1 - \phi_n}{\phi_n}} f_2(kz^{\chi_{nk}}, nz) dz + (1 - \phi_n)R, \end{aligned}$$

where, for notational simplicity,  $f_1(kz, nz^{\chi_{kn}}) = \partial f(kz, nz^{\chi_{kn}})/\partial(kz)$  and  $f_2(kz^{\chi_{nk}}, nz) = \partial f(kz^{\chi_{nk}}, nz)/\partial(nz)$ . As in the basic model, the price of capital is a weighted average of infra-marginal productivities with respect to capital over the relevant range  $\tilde{\delta}k$  (in case of no investment) to  $k$  (actual investment). Similarly, the wage rate is a weighted average of infra-marginal productivities with respect to labor (since wages for all employees, incumbent and new, are renegotiated in every period, the relevant range extends from 0 to  $n$ ). Using these expressions, capital demand in (32) and the capital price in (34) can be expressed as

$$\rho(k, n) = OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1 - \delta)s_k\varphi}{\tilde{\delta}} - \frac{\gamma_k}{\beta p(\theta_k)} \quad (36)$$

$$\rho(k, n) = \phi_k \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1 - \delta)s_k\varphi}{\tilde{\delta}} \right] + (1 - \phi_k)\zeta, \quad (37)$$

with the 'overinvestment factor' being defined as

$$OI = 1 - \frac{\tilde{\delta}k \int_{\tilde{\delta}}^1 z^{\frac{1}{\tilde{\delta}\tilde{\phi}_k}} \frac{f_{11}(kz, nz^{\chi_{kn}})}{\tilde{\delta}} dz + n \int_0^1 z^{\frac{1 - \phi_n}{\phi_n}} \left(1 + \frac{\tilde{\delta}\tilde{\phi}_k}{1 - \tilde{\delta}\tilde{\phi}_k}\right) \frac{f_{12}(kz^{\chi_{nk}}, nz)}{\tilde{\delta}} dz}{\frac{\partial f(k, n)/\partial k}{\tilde{\delta}}}.$$



Likewise, labor demand in (33) and the wage rate in (35) become

$$w(k, n) = OE \times \partial f(k, n) / \partial n - \frac{\gamma_n(r + s_n)}{p(\theta_n)} \quad (38)$$

$$w(k, n) = \phi_n [OE \times \partial f(k, n) / \partial n] + (1 - \phi_n)R, \quad (39)$$

with the 'overemployment factor' being defined as

$$OE = 1 - \frac{\tilde{\delta}k \int_{\tilde{\delta}}^1 z^{\frac{1-\tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k}} \left(1 + \frac{\phi_n}{1-\phi_n}\right) f_{12}(kz, nz^{\chi_{kn}}) dz + n \int_0^1 z^{\frac{1}{\phi_n}} f_{22}(kz^{\chi_{nk}}, nz) dz}{\partial f(k, n) / \partial n}.$$

Consider the overinvestment factor  $OI$ . As in the basic model, decreasing marginal productivity of capital (i.e.  $f_{11} < 0$ ) pushes firms to overinvest in order to reduce the holdup problem in the capital market. At the same time, the firm also needs to take into account that capital affects the marginal productivity of labor and thus the holdup problem in the labor market. If capital and labor are substitutes (i.e.  $f_{12} < 0$ ), a larger capital stock also lowers the marginal productivity of labor and thus reinforces the overinvestment tendency. If, by contrast, capital and labor are complements (i.e.  $f_{12} > 0$ ), then the firm faces a trade-off because a larger capital stock worsens the holdup problem in the labor market. In this case, holdup problems in the labor market reduce the overinvestment motive of the firm and, depending on the details of technology and the bargaining power of suppliers relative to workers, may even result in underinvestment. A similar analysis applies for the overemployment factor  $OE$ . The following proposition summarizes these consequences.

**Proposition 9.** *If capital and labor are substitutes and  $f(k, n)$  is strictly concave in both arguments over some segment of  $[(1 - \delta)k, k]$  and  $[0, n]$ , respectively, the two holdup problems lead to both overinvestment and overemployment. If capital and labor are complements, the two holdup problems counteract each other, reducing the tendency to both overinvest and overemploy.*

The proposition offers a cautionary tale that holdup problems in the labor market do not necessarily lead to underinvestment, as is typically emphasized in the literature (e.g. Grout, 1984; Acemoglu and Shimer, 1999). While a more detailed analysis with different types of labor and capital is beyond the scope of this paper, this seems to be especially relevant for capital goods that substitute for types of labor with strong bargaining power (e.g. relatively low-skilled but highly unionized labor). Furthermore, the proposition illustrates that multiple holdup problems do not

need to exacerbate each other as is often the case in the literature (e.g. Aruoba, Waller and Wright, 2008). Rather, if the factors subject to holdup problems are complementary for allocations, then the decentralized economy may be relatively close to efficiency even if each holdup problem on its own severely distorts allocations.

Finally, the decentralized equilibrium can be computed by combining (30) and (36) with (37) to eliminate the price of capital; and (38) with (39) to eliminate the wage rate. The following proposition describes this equilibrium.

**Proposition 10.** *There exists a unique decentralized bargaining equilibrium  $(k^B, \theta_k^B, n^B, \theta_n^B)$  that solves*

$$\gamma_k = p(\theta_k)(1 - \phi_k)\beta \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1 - \delta)s_k\varphi}{\tilde{\delta}} - \zeta \right] \quad (40)$$

$$1 - \beta\zeta = q(\theta_k)\phi_k\beta \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1 - \delta)s_k\varphi}{\tilde{\delta}} - \zeta \right] \quad (41)$$

$$\gamma_n = p(\theta_n)(1 - \phi_n)\beta \left[ \frac{OE \times \partial f(k, n)/\partial n - b}{1 - [1 - s_n - \phi_n q(\theta_n)(1 - s_n)]\beta} \right] \quad (42)$$

$$n = \frac{q(\theta_n)}{s_n + (1 - s_n)q(\theta_n)} \quad (43)$$

*This equilibrium is generally inefficient.*

PROOF: Appendix B.

The system of equations in (40)-(43) differs from their efficient counterpart in (26)-(29) in two respects: the presence of the overinvestment and overemployment due to the double holdup problem; and the difference in how the respective surpluses are split due to the matching externalities. In particular, recall that  $m_{v_k} = p(\theta_k)(1 - \epsilon(\theta_k))$ ,  $m_l = q(\theta_k)\epsilon(\theta_k)$  and  $m_{v_n} = p(\theta_n)(1 - \epsilon(\theta_n))$ . Hence, Hosios' (1990) condition is satisfied in the capital market and the labor market if and only if  $\phi_k = \epsilon(\theta_k^S)$  and  $\phi_n = \epsilon(\theta_n^S)$ , respectively. But in this case, it is still generally true that the holdup problems do not cancel each other out and thus,  $OI \neq 1$  and  $OE \neq 1$ .<sup>28</sup>

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<sup>28</sup>As long as capital and labor are complements, there may exist knife-edge cases where the bargaining powers satisfy Hosios' condition in each market and, simultaneously, the two holdup problems cancel each other out. The decentralized economy thus attains efficiency. It is straightforward to show that the Cobb-Douglas production function contains such a knife-edge case.

## 6 Conclusion

The paper shows that two characteristics of physical capital – ex-post specificity of capital goods and sequentiality of investment decisions – can lead to a potentially important holdup problem that has previously been neglected in the literature. Under the assumption that different capital inputs are substitutes, this holdup problem leads to overinvestment in partial equilibrium. This result contrasts with much of the literature. The difference is due to particular assumptions about agents and markets, thus highlighting the crucial role of microeconomic structure for the analysis of holdup problems.<sup>29</sup>

In general equilibrium, overinvestment interacts with the trading frictions behind specificity as well as other holdup problems in factor markets. An application in a general equilibrium macro context with holdups in both physical capital and labor markets shows that the resulting allocations are generally inefficient. Whether the overinvestment result prevails in such a general equilibrium setting and what the welfare losses from these frictions are remain quantitative questions that exceed the scope of this paper.<sup>30</sup> Yet, the policy implications of overinvestment are clearly important. In particular, overinvestment provides a rationale for capital income taxation or, at the least, counteracts other forces that, on their own, imply underinvestment and capital income subsidies (e.g. Aruoba and Chugh, 2008).<sup>31</sup>

Overinvestment due to holdup problems may also help explain a number of empirical phenomena. On a macroeconomic level, Caballero and Hammour (1998b) argue that worsening holdups of labor on capital and factor substitutability provide an explanation for the sustained increase of the capital-labor ratio in different European countries from the 1970s through the 1990s. The analysis in Section 5 suggests that overinvestment due to holdup problems in capital markets amplifies these effects. On a microeconomic level, holdup problems in physical capital markets may explain why

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<sup>29</sup>De Meza and Lockwood (2004, 2007) explore alternative mechanisms due to coordination failure and heterogeneity across agents that imply overinvestment as a result of holdup problems. Other examples outside of the holdup literature that, under some conditions, generate overinvestment are Chien and Lee (2008) or Lagos and Rocheteau (2008a).

<sup>30</sup>Preliminary calculations in a model calibrated to the U.S. economy suggest that the welfare losses can easily exceed 2-3% in consumption equivalents. The importance of these losses should not be surprising as distortions of intertemporal margins usually have large effects. See for example Aruoba, Waller and Wright (2008) for results of similar size.

<sup>31</sup>Arseneau, Chugh and Kurmann (2008) show that quasi-rents arising from trading frictions in physical capital markets may, on their own, provide a motive for capital income taxation.

we observe suppliers for large capital expenditure projects enter into a consortium that negotiates as a single party with the firm. By doing so, the suppliers can eliminate the firm's strategic incentive to overinvest that arises from pairwise negotiation within a time period. On a more general level, specificity and holdup problems in physical capital markets may offer new interesting insights about firm dynamics. In an environment with entry, strategic investment motives could provide an alternative explanation for why firms grow gradually and in particular, why firm growth decreases with size. Finally, the idea that capital allocation is subject to trading frictions that vary with the degree of specificity of capital goods may help explain persistent differences in firm investment behavior and financial performance across industries.

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## A Derivations for the basic model

### A.1 Proof of Proposition 1

The social planner's problem is

$$\begin{aligned} O(k, u) &= \max_{v, l} [f(k) + \zeta u - \gamma v - l + O(k_{+1}, u_{+1})] \\ \text{s.t. } k_{+1} &= (1 - \delta)k + m(v, l) \\ u_{+1} &= l - m(v, l). \end{aligned}$$

The first-order conditions (expressed in steady state) are

$$\gamma = m_v [O_k - O_u] \quad (44)$$

$$1 = m_l O_k + (1 - m_l) O_u \quad (45)$$

and the corresponding envelope conditions are

$$O_u = \zeta \quad (46)$$

$$O_k = f'(k) + (1 - \delta) O_k. \quad (47)$$

Using the envelope conditions to substitute out for the marginal values in equations (44) and (45) yields

$$\gamma = m_v \left[ \frac{f'(k)}{\delta} - \zeta \right] \quad (48)$$

$$1 - \zeta = m_l \left[ \frac{f'(k)}{\delta} - \zeta \right]. \quad (49)$$

These two equations are (3) and (4) in the main text. To compute the equilibrium, combine the two equations to substitute out the expression in brackets. This yields

$$\frac{\gamma}{1 - \zeta} = \frac{m_v}{m_l}.$$

But since  $m_v = (1 - \epsilon(\theta))p(\theta)$  and  $m_l = \epsilon(\theta)q(\theta) = \epsilon(\theta)p(\theta)\theta$ , this expression can be rewritten as

$$\theta^S = \frac{1 - \zeta}{\gamma} \frac{1 - \epsilon(\theta)}{\epsilon(\theta)}. \quad (50)$$

Under the conditions imposed on the matching function,  $\epsilon(\theta)$  is monotonically increasing in  $\theta$  with  $\lim_{\theta \rightarrow 0} \epsilon(\theta) = 0$  and  $\lim_{\theta \rightarrow \infty} \epsilon(\theta) = 1$ . Hence, there exists a unique efficient steady state capital market tightness  $\theta^S = [0, \infty]$ . In turn, the equilibrium capital stock  $k^S$  is determined by (44), which can be rewritten as

$$\frac{f'(k)}{\delta} = \frac{\gamma}{m_v} + \zeta.$$

Given the properties of the production function, the left-hand side is a monotonically decreasing in  $k$  with  $\lim_{k \rightarrow 0} f'(k)/\delta = \infty$  and  $\lim_{k \rightarrow \infty} f'(k)/\delta = 0$ . Concurrently, the right-hand side is a monotonically



increasing function of  $\theta$  with  $\lim_{\theta \rightarrow 0} \left( \frac{\gamma}{m_v} + \zeta \right) = \gamma + \zeta$  and  $\lim_{\theta \rightarrow \infty} \left( \frac{\gamma}{m_v} + \zeta \right) = \infty$  by the regularity conditions imposed on the matching function. Hence, (44) defines a schedule for  $(\theta, k)$  that starts at  $k = f'^{-1}(\delta\gamma/m_v + \delta\zeta) > 0$  for  $\theta = 0$  and then decreases monotonically to  $k = 0$  as  $\theta \rightarrow \infty$ . Together with (50), this schedule implies a unique equilibrium  $k^S \in [0, f'^{-1}(\delta\gamma/m_v + \delta\zeta)]$ , as illustrated in Figure 1 of the main text. ■

## A.2 Proof of Propositions 2 and 3

To derive the equilibrium equations of the decentralized model with Nash bargaining, combine the optimal entry condition for the capital supplier (6) with the free-entry condition. In steady state, this yields

$$1 - \zeta = q(\theta)(\rho - \zeta). \quad (51)$$

On the firm side, combine (7) with (8) to obtain, in steady state,

$$\gamma = p(\theta) \left[ \frac{f'(k) - \rho'(k)\delta k}{\delta} - \rho(k) \right]. \quad (52)$$

This expression can be rearranged to obtain the capital demand equation (10) in the main text

$$\rho(k) = \frac{f'(k) - \rho'(k)\delta k}{\delta} - \frac{\gamma}{p(\theta)}. \quad (53)$$

Next, use the generalized Nash bargaining solution  $\phi \partial J / \partial k = (1 - \phi)(W_k - W_u)$  and combine it with the appropriate marginal values to obtain the price of capital

$$\begin{aligned} \phi \left[ \frac{f'(k) - \rho'(k)\delta k}{\delta} - \rho(k) \right] &= (1 - \phi)(\rho(k) - \zeta) \\ \Leftrightarrow \rho(k) &= \phi \left[ \frac{f'(k) - \rho'(k)\delta k}{\delta} \right] + (1 - \phi)\zeta. \end{aligned} \quad (54)$$

This is a non-homogenous linear differential equation of  $\rho$  in  $k$ . Using standard tools from differential calculus, the solution for the homogenous part of this equation,  $\rho^h(k) = \phi [f'(k)/\delta - \rho'(k)k]$ , is

$$\rho^h(k) = k^{-\frac{1}{\phi}} \left[ \int_{(1-\delta)k}^k z^{\frac{1-\phi}{\phi}} \frac{f'(z)}{\delta} dz + D \right].$$

The relevant support of integration in this expression is  $[(1 - \delta)k, k]$  because in every period, the firm's possible range of investments extends from 0 (no investment in which case the capital stock would reduce to  $(1 - \delta)k$ ) to  $\delta k$  (the actual investment such that the capital stock remains at  $k$ ).  $D$  is the constant of integration. Under the assumption that  $\lim_{\kappa \rightarrow (1-\delta)k} [\kappa - (1 - \delta)k] \rho(\kappa) = 0$ , this constant needs to be

$D = 0$ .<sup>32</sup> Hence, equation (54) becomes

$$\rho(k) = k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1-\phi}{\phi}} \frac{f'(z)}{\delta} dz + (1-\phi)\zeta,$$

which is equation (11) in the main text. One can rewrite  $k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1-\phi}{\phi}} \frac{f'(z)}{\delta} dz = \int_0^1 z^{\frac{1-\phi}{\phi}} \frac{f'(kz)}{\delta} dz$ . This, in turn, implies  $\rho'(k)k = k \int_{(1-\delta)k}^1 z^{\frac{1-\phi}{\phi}} z \frac{f''(kz)}{\delta} dz = k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1}{\phi}} \frac{f''(z)}{\delta} dz$ . Hence, capital demand in (53) and the price of capital in (54) can be expressed as

$$\rho(k) = OI \times \frac{f'(k)}{\delta} - \frac{\gamma}{p(\theta)} \quad (55)$$

$$\rho(k) = \phi \left[ OI \times \frac{f'(k)}{\delta} \right] + (1-\phi)\zeta, \quad (56)$$

with  $OI \equiv \left[ 1 - \frac{k^{-\frac{1}{\phi}} \int_{(1-\delta)k}^k z^{\frac{1}{\phi}} f''(z) dz}{f'(k)} \right]$  denoting the 'overinvestment factor'. For  $f''(\cdot) \leq 0$  with strict inequality over some segment of  $((1-\delta)k, k)$ , the integral in the definition of  $OI$  is negative and thus  $OI > 1$ . Hence, for a given price of capital, the firm accumulates more capital than is warranted by the marginal productivity  $f'(k)$ ; i.e. the firm overinvests. This proves Proposition 2. ■

To compare the decentralized equilibrium to the efficient allocation, take the capital demand equation in (55) and the capital supply equation in (51) and use the solution for the price of capital to obtain, respectively

$$\frac{\gamma}{p(\theta)} = (1-\phi) \left[ OI \times \frac{f'(k)}{\delta} - \zeta \right] \quad (57)$$

$$\frac{1-\zeta}{q(\theta)} = \phi \left[ OI \times \frac{f'(k)}{\delta} - \zeta \right]. \quad (58)$$

These are equations (13) and (14) in the main text. Combining the two equations to eliminate the bracketed expression yields the equilibrium condition for  $\theta$

$$\theta^B = \frac{1-\zeta}{\gamma} \frac{1-\phi}{\phi}. \quad (59)$$

Hence, there exists a unique equilibrium value  $\theta^B \in [0, \infty]$ . Furthermore, rearrange (57) as  $OI \times \frac{f'(k)}{\delta} = \frac{1}{1-\phi} \frac{\gamma}{p(\theta)} + \zeta$ . The above properties of  $\rho'(k)k$  imply that the left-hand side is a monotonically decreasing function of  $k$  with  $\lim_{k \rightarrow 0} OI \times f'(k)/\delta = \infty$  and  $\lim_{k \rightarrow \infty} OI \times f'(k)/\delta = 0$ . Concurrently, the right-hand side is a monotonically increasing function of  $\theta$  with  $\lim_{\theta \rightarrow 0} \left( \frac{1}{1-\phi} \frac{\gamma}{p(\theta)} + \zeta \right) = \frac{\gamma}{1-\phi} + \zeta$  and  $\lim_{\theta \rightarrow \infty} \left( \frac{1}{1-\phi} \frac{\gamma}{p(\theta)} + \zeta \right) = \infty$ . Hence, the expression defines a schedule for  $(\theta, k)$  that starts at  $k = f'^{-1} \left( \frac{\frac{\delta}{1-\phi} \frac{\gamma}{p(\theta)} + \delta \zeta}{OI} \right) > 0$  for  $\theta = 0$  and then decreases monotonically to  $k = 0$  as  $\theta \rightarrow \infty$ . Together

<sup>32</sup>See Appendix B of Cahuc, Marque and Wasmer (2007) for a more rigorous description of the conditions under which the solution to that differential equation is well defined.

with (59), this schedule implies that there exists a unique equilibrium capital stock  $k^B$ .

To prove the second part of Proposition 3, note that  $\theta^B = \theta^S$  if and only if  $\phi = \epsilon(\theta^S)$ . In that case, Hosios's condition is satisfied and there is no externality from matching. A comparison of (48) with (57) shows that at this point  $k^B > k^S$  because  $OI \times f'(k) > f'(k)$  for any given  $k$ . This proves that the decentralized equilibrium is never efficient. Now, consider the case  $\phi > \epsilon(\theta^S)$ . This implies  $\theta^B < \theta^S$  and thus  $p(\theta^B) > p(\theta^S)$ . Hence, the decentralized solution  $\frac{f'(k)}{\delta} = \left(\frac{1}{1-\phi} \frac{\gamma}{p(\theta^B)} + \zeta\right) / OI$  may be smaller or larger than the efficient solution  $\frac{f'(k)}{\delta} = \left(\frac{\gamma}{m_v} + \zeta\right) = \left(\frac{1}{1-\epsilon(\theta^S)} \frac{\gamma}{p(\theta^S)} + \zeta\right)$ , which means that  $k^B \lesseqgtr k^S$  depending on which effect prevails. For the opposite case  $\phi < \epsilon(\theta^S)$ , we have  $\theta^B > \theta^S$  and thus, a similar analysis applies. In general,  $k^B < k^S$  if the externality from the matching friction is sufficiently large to overwhelm the overinvestment effect. This completes the proof of Proposition 3. ■

### A.3 Proof of Proposition 4

The firm's problem is

$$\begin{aligned} J(k, k_{-1}) &= \max_{v, \rho} [f(k) - \rho[k - (1 - \delta)k_{-1}] - \gamma v + J(k_{+1}, k)] \\ &\quad s.t. \ k_{+1} = (1 - \delta)k + p(\theta)v \\ &\quad s.t. \ 1 - \zeta = q(\theta)(\rho - \zeta) \end{aligned}$$

The first-order condition with respect to  $v$  and the associated envelope condition for  $k$  are identical to the ones in the Nash-bargaining case in (7) and (8). Together, they define the optimal venture posting condition

$$\gamma = p(\theta) \left[ \frac{f'(k)}{\delta} - \rho \right] \quad (60)$$

Simultaneously, the firm sets the posted price such as to maximize the value of optimal venture postings in (60) subject to the suppliers' queuing condition. The first-order condition is

$$0 = p'(\theta) \frac{\partial \theta}{\partial \rho} \left[ \frac{f'(k)}{\delta} - \rho \right] - p(\theta)$$

To find  $\partial \theta / \partial \rho$ , compute the partial derivative of (16) with respect to  $\rho$  and rearrange

$$\frac{\partial \theta}{\partial \rho} = -\frac{q(\theta)}{q'(\theta)(\rho - \zeta)}.$$

Using this equations to substitute out  $\partial \theta / \partial \rho$  from the above condition for optimal  $\rho$  yields

$$p(\theta)q'(\theta)(\rho - \zeta) = -p'(\theta)q(\theta) \left[ \frac{f'(k)}{\delta} - \rho \right],$$

or equivalently

$$\rho = \epsilon(\theta) \frac{f'(k)}{\delta} + (1 - \epsilon(\theta))\zeta,$$

where I used the fact that  $q'(\theta)\theta/q(\theta) = 1 - \epsilon(\theta)$  and  $-p'(\theta)\theta/p(\theta) = \epsilon(\theta)$ . Plugging this solution into

(16) and (60) to eliminate  $\rho$  yields, respectively

$$\begin{aligned} 1 - \zeta &= q(\theta)\epsilon(\theta) \left[ \frac{f'(k)}{\delta} - \zeta \right] \\ \gamma &= p(\theta)(1 - \epsilon(\theta)) \left[ \frac{f'(k)}{\delta} - \zeta \right]. \end{aligned}$$

Dividing the two equations to eliminate the bracketed expression then implies a unique equilibrium capital market tightness  $\theta^P \in [0, \infty]$

$$\theta^P = \frac{1 - \zeta}{\gamma} \frac{1 - \epsilon(\theta)}{\epsilon(\theta)}.$$

The optimal venture posting condition in (60) then implies the equilibrium capital stock

$$\frac{f'(k)}{\delta} = \frac{\gamma}{p(\theta)(1 - \epsilon(\theta))} + \zeta.$$

Since  $p(\theta)(1 - \epsilon(\theta)) = m_v$ , this equilibrium is equivalent to the social planner equilibrium, thus establishing efficiency. This proves Proposition 4. ■

#### A.4 Proof of Proposition 5

This part analyzes the limiting case where posting costs  $\gamma$  are reduced to 0. Consider first the efficient allocation. Condition (48) implies that  $m_v = 0$  for  $\gamma = 0$  for arbitrary match surpluses. But by the regularity conditions of the matching function,  $m_v = 0$  implies  $\theta = \infty$ ; i.e. when venture costs are zero, the planner naturally posts an infinity of ventures. This, in turn, means that  $m_l = 1$  and thus, (49) becomes  $1 - \zeta = [f'(k)/\delta - \zeta]$ , or equivalently

$$1 = f'(k) + 1 - \delta. \tag{61}$$

For the decentralized equilibrium, (59) implies that  $\gamma\theta$  is finite and that  $\theta = \infty$  for  $\gamma = 0$ . By the regularity conditions of the matching function, we thus have  $p(\theta) = 0$  and  $q(\theta) = 1$ . Hence, (51) implies  $\rho = 1$ . Furthermore, since  $p(\theta)$  exhibits decreasing returns to scale in  $\theta$  by definition of the matching function,  $\gamma/p(\theta) \rightarrow 0$  as  $\gamma \rightarrow 0$ . But then, the capital demand in (55) implies that  $\rho = 1 = OI \times f'(k)/\delta$ , or equivalently

$$1 = OI \times f'(k) + 1 - \delta.$$

As long as  $OI > 1$ , we thus have  $k^B > k^S$ , which implies that the decentralized equilibrium is inefficient. This proves Proposition 5. ■

#### A.5 Proof of Proposition 6

Consider first a change in  $\gamma$ . By (59), an increase in  $\gamma$  decreases equilibrium capital market tightness  $\theta^B$  proportionally such that  $\gamma\theta^B$  is constant. Then, use (57) to rewrite it as

$$OI \times \frac{f'(k)}{\delta} = \frac{1}{1 - \phi} \frac{\gamma}{p(\theta)} + \zeta = \frac{1}{1 - \phi} \frac{\gamma\theta}{q(\theta)} + \zeta.$$

The right-hand side is increasing in  $\gamma$  since  $\gamma\theta$  is constant and  $q(\theta)$  is decreasing in  $\gamma$ . The left-hand side is decreasing in  $k$ . Hence, an increase in  $\gamma$  implies a decrease in  $k$ .

Now, consider a change in  $\zeta$ . By (59), an increase in  $\zeta$  also decreases equilibrium capital market tightness  $\theta^B$ . However, by (57), the effect on  $k$  is ambiguous because an increase in  $\zeta$  both increases the right-hand side (directly through  $\zeta$ ) and decreases it (through the accompanying increase in  $p(\theta)$ ). This proves Proposition 6. ■

## A.6 Proof of Proposition 7

To derive the decentralized equilibrium, the model is solved backwards. In the second stage,  $k_1$  is taken as given and the first-order conditions of the Nash bargaining problem  $\max_{k_2, \rho_2} J^{1-\phi} W_2^\phi$  are

$$\begin{aligned} (1 - \phi)W_2 &= \phi J \\ (1 - \phi)W_2[f_{k_2} - \rho_2] &= -\phi J[\rho_2 - c_{k_2}]. \end{aligned}$$

Combining the two conditions yields

$$\frac{\partial f}{\partial k_2} = \frac{\partial c(k_2)}{\partial k_2}.$$

Using the implicit function theorem, it is possible to derive that

$$\frac{\partial k_2}{\partial k_1} = -\frac{\partial^2 f / (\partial k_1 \partial k_2)}{\partial^2 f / (\partial k_2 \partial k_2) - \partial^2 c / (\partial k_1 \partial k_2)}.$$

Since  $\partial^2 f / (\partial k_2 \partial k_2) \leq 0$  and  $\partial^2 c / (\partial k_1 \partial k_2) \geq 0$  by assumption of the model,  $\partial k_2 / \partial k_1 < 0$  iff  $\partial^2 f / (\partial k_1 \partial k_2) < 0$  (i.e.  $k_1$  and  $k_2$  are substitutes) and  $\partial k_2 / \partial k_1 > 0$  iff  $\partial^2 f / (\partial k_1 \partial k_2) > 0$  (i.e.  $k_1$  and  $k_2$  are complements). Next, using the definitions of  $J$  and  $W$  in the main text, the first condition above yields an explicit solution for  $\rho_2$

$$\rho_2 = \phi \frac{f(k_1, k_2) - \rho_1 k_1}{k_2} + (1 - \phi) \frac{c(k_2)}{k_2},$$

which implies that

$$\begin{aligned} \frac{\partial \rho_2}{\partial k_1} k_2 &= \phi \left[ \frac{\partial f}{\partial k_1} - \rho_1 \right] + \phi \left[ \frac{\partial f}{\partial k_2} - \frac{f(k_1, k_2) - \rho_1 k_1}{k_2} \right] \frac{\partial k_2}{\partial k_1} \\ &\quad + (1 - \phi) \left[ c_{k_2} - \frac{c(k_2)}{k_2} \right] \frac{\partial k_2}{\partial k_1}. \end{aligned}$$

In the first stage, the demand for  $k_1$  is given by

$$\rho_1 = \frac{\partial f}{\partial k_1} - \left[ \frac{\partial \rho_2}{\partial k_1} k_2 + \rho_2 \frac{\partial k_2}{\partial k_1} \right].$$

Using the derivations from the second stage, the expression in brackets is

$$\begin{aligned} \frac{\partial \rho_2}{\partial k_1} k_2 + \rho_2 \frac{\partial k_2}{\partial k_1} &= \phi \left[ \frac{\partial f}{\partial k_1} - \rho_1 \right] + \phi \left[ \frac{\partial f}{\partial k_2} - \frac{f(k_1, k_2) - \rho_1 k_1}{k_2} \right] \frac{\partial k_2}{\partial k_1} \\ &\quad + (1 - \phi) \left[ \frac{\partial c(k_2)}{\partial k_2} - \frac{c(k_2)}{k_2} \right] \frac{\partial k_2}{\partial k_1} + \left[ \phi \frac{f(k_1, k_2) - \rho_1 k_1}{k_2} + (1 - \phi) \frac{c(k_2)}{k_2} \right] \frac{\partial k_2}{\partial k_1}. \end{aligned}$$

or equivalently,

$$(1 - \phi) \left[ \frac{\partial \rho_2}{\partial k_1} k_2 + \rho_2 \frac{\partial k_2}{\partial k_1} \right] = \left[ \phi \frac{\partial f}{\partial k_2} + (1 - \phi) \frac{\partial c(k_2)}{\partial k_2} \right] \frac{\partial k_2}{\partial k_1}.$$

This implies that  $\frac{\partial \rho_2}{\partial k_1} k_2 + \rho_2 \frac{\partial k_2}{\partial k_1} \leq 0$  and thus  $\frac{\partial c(k_1)}{\partial k_1} = \rho_1 \geq \frac{\partial f}{\partial k_1}$  iff  $\partial k_2 / \partial k_1 \leq 0$ . From the derivations in the second stage, we know that  $\partial k_2 / \partial k_1 < 0$  if  $\partial^2 f / (\partial k_1 \partial k_2) < 0$ , in which case  $k_1^B > k_1^S$  (overinvestment). Vice versa,  $\partial k_2 / \partial k_1 > 0$  if  $\partial^2 f / (\partial k_1 \partial k_2) > 0$ , in which case  $k_1^B < k_1^S$  (underinvestment). This proves Proposition 7. ■

## B Derivations of model with holdups in labor market

### B.1 Proof of Proposition 8

The social planner's problem is

$$\begin{aligned} O(k, u_k, n) &= \max_{c, l, k, v_k, v_n} [\log c + \beta O(k_{+1}, u_{k,+1}, n_{+1})] \\ &\quad + \lambda [f(k, n) + b(1 - n) + (1 - \delta) s_k \varphi k + \zeta u_k - \gamma_k v_k - \gamma_n v_n - c - l] \\ \text{s.t. } k_{+1} &= (1 - \delta)(1 - s_k)k + m(v_k, l) \\ u_{k,+1} &= l - m(v_k, l) \\ n_{+1} &= (1 - s_n)n + m(v_n, 1 - n + s_n n). \end{aligned}$$

In steady state, the first-order conditions of the optimization problem are

$$\begin{aligned} \frac{1}{c} &= \lambda \\ \beta \left[ m_l \frac{\partial O}{\partial k} + (1 - m_l) \frac{\partial O}{\partial u_k} \right] &= \lambda \\ m_{v_k} \beta \left[ \frac{\partial O}{\partial k} - \frac{\partial O}{\partial u_k} \right] &= \lambda \gamma_k \\ m_{v_n} \beta \frac{\partial O}{\partial n} &= \lambda \gamma_n \end{aligned}$$

and the corresponding envelope conditions are

$$\begin{aligned}
\frac{\partial O}{\partial u_k} &= \lambda \zeta \\
\frac{\partial O}{\partial k} &= \lambda \left[ \frac{\partial f}{\partial k} + (1 - \delta)s_k \varphi \right] + (1 - \delta)(1 - s_k)\beta \frac{\partial O}{\partial k} \\
\frac{\partial O}{\partial n} &= \lambda \left[ \frac{\partial f}{\partial n} - b \right] + [1 - s_n - m_{u_n}(1 - s_n)]\beta \frac{\partial O}{\partial n}.
\end{aligned}$$

Combining the second and third first-order conditions with the corresponding envelope conditions yields two conditions describing liquid capital supply and capital demand

$$1 - \beta \zeta = m_l \beta \left[ \frac{f_k + (1 - \delta)s_k \varphi}{1 - (1 - \delta)(1 - s_k)\beta} - \zeta \right] \quad (62)$$

$$\gamma_k = m_{v_k} \beta \left[ \frac{f_k + (1 - \delta)s_k \varphi}{1 - (1 - \delta)(1 - s_k)\beta} - \zeta \right] \quad (63)$$

Note that the opportunity cost of capital  $\lambda$  drops out of these equations and thus, the form of consumption preferences does not directly influence capital allocation. On the labor market side, combining the last first-order condition with the last envelope condition yields

$$\frac{\partial f}{\partial n} - b = \frac{\gamma_n}{1 - \epsilon(\theta_n)} \left[ \frac{r + s_n}{p(\theta_n)} + (1 - s_n)\epsilon(\theta_n)\theta_n \right] \quad (64)$$

where  $1 - \epsilon(\theta_n) = \partial m(v_n, 1 - n + s_n n) / \partial v_n \times v_n / m(v_n, 1 - n + s_n n) = m_{v_n} / p(\theta_n)$ , and  $\epsilon(\theta_n) = \partial m(v_n, 1 - n + s_n n) / \partial (1 - n + s_n n) \times (1 - n + s_n n) / m(v_n, 1 - n + s_n n) = m_{u_n} / q(\theta_n)$ .

To derive the equilibrium, divide (62) by (63) to obtain

$$\theta_k^S = \frac{1 - \beta \zeta}{\gamma_k} \frac{1 - \epsilon(\theta_k)}{\epsilon(\theta_k)}. \quad (65)$$

where  $1 - \epsilon(\theta_k) = \partial m(v_k, u_k) / \partial v_k \times v_k / m(v_k, u_k) = m_{v_k} / p(\theta_k)$  and  $\epsilon(\theta_k) = \partial m(v_k, u_k) / \partial u_k \times u_k / m(v_k, u_k) = m_{u_k} / q(\theta_k)$ . Hence, there exists a unique equilibrium  $\theta_k^S \in [0, \infty]$ . Equation (63), in turn, can be reexpressed as

$$\frac{\partial f}{\partial k} = \frac{r + \delta + s_k(1 - \delta)}{1 + r} \zeta - (1 - \delta)s_k \varphi + \frac{\gamma_k}{1 - \epsilon(\theta_k)} \frac{r + \delta + s_k(1 - \delta)}{p(\theta_k)}.$$

This pins down the optimal capital-labor ratio  $(k/n)^S$  since for any CRTS production function  $f(k, n)$ , its partial derivatives depend on  $k/n$  only. Given  $k/n$ , (64) yields the optimal labor market tightness  $\theta_n^S$ . This, in turn, determines equilibrium employment  $n^S$  by the steady state law of motion for employment

$$n^S = \frac{q(\theta_n)}{s_n + (1 - s_n)q(\theta_n)},$$

and thus the optimal matched capital stock  $k^S$ . This completes the proof of Proposition 8. ■

## B.2 Proof of Propositions 9 and 10

The representative household's problem is

$$\begin{aligned}
V(i, u_k, n) &= \max_{c, l} \log c + \beta V(i_{+1}, u_{k,+1}, n_{+1}) \\
&\quad + \lambda [wn + b(1 - n) + \rho i + \zeta u_k + d - c - l] \\
\text{s.t. } i_{+1} &= q(\theta_k)l \\
u_{k,+1} &= (1 - q(\theta_k))l \\
n_{+1} &= (1 - s_n)n + q(\theta_n)(1 - n + s_n n)
\end{aligned}$$

where  $d$  are firm profits transferred lump-sum to households. In steady state, the first-order conditions of the optimization problem are

$$\begin{aligned}
\frac{1}{c} &= \lambda \\
\lambda &= \beta \left[ q(\theta_k) \frac{\partial V}{\partial i} + (1 - q(\theta_k)) \frac{\partial V}{\partial u_k} \right]
\end{aligned}$$

and the envelope conditions are

$$\begin{aligned}
\frac{\partial V}{\partial u_k} &= \lambda \zeta \\
\frac{\partial V}{\partial i} &= \lambda \rho \\
\frac{\partial V}{\partial n} &= \lambda [w - b] + [1 - s_n - (1 - s_n)q(\theta_n)] \beta \frac{\partial V}{\partial n}
\end{aligned}$$

Note that this last equation is the difference in value from being employed relative to being unemployed. Alternatively, one could have derived this envelope condition as is done in most of the labor search literature by defining the value of employment as

$$E = \lambda w + [1 - s_n(1 - q(\theta_n))] \beta E + s_n(1 - q(\theta_n)) \beta U$$

and the value of unemployment as

$$U = \lambda b + q(\theta_n) \beta E + (1 - q(\theta_n)) \beta U.$$

The difference between the two values equals

$$E - U = \lambda(w - b) + [1 - s_n - (1 - s_n)q(\theta_n)] \beta (E - U),$$

which is equivalent to  $\partial V / \partial n$  above. In steady state, the envelope conditions for the capital side together with the optimal choice for liquid capital yields

$$1 - \beta \zeta = q(\theta_k) \beta [\rho - \zeta]. \tag{66}$$



The representative firm's problem is

$$\begin{aligned}
J(k, k_{-1}, n) &= \max_{v_k, v_n} \left[ \begin{aligned} &f(k, n) - \rho(k, n)[k - (1 - \delta)(1 - s_k)k_{-1}] + s_k(1 - \delta)\varphi k \\ &- w(k, n)n - \gamma_k v_k - \gamma_n v_n + \beta J(k_{+1}, k, n_{+1}) \end{aligned} \right] \\
\text{s.t. } k_{+1} &= (1 - \delta)(1 - s_k)k + p(\theta_k)v_k \\
n_{+1} &= (1 - s_n)n + p(\theta_n)v_n.
\end{aligned}$$

In steady state, the first-order conditions of the optimization problem are

$$\begin{aligned}
\gamma_k &= p(\theta_k)\beta \frac{\partial J}{\partial k_{+1}} \\
\gamma_n &= p(\theta_n)\beta \frac{\partial J}{\partial n}
\end{aligned}$$

and the envelope conditions are

$$\begin{aligned}
\frac{\partial J}{\partial k_{+1}} &= \frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} [1 - (1 - \delta)(1 - s_k)]k - \rho + \rho(1 - \delta)(1 - s_k)\beta \\
&\quad + s_k(1 - \delta)\varphi - \frac{\partial w}{\partial k}n + (1 - \delta)(1 - s_k)\beta \frac{\partial J}{\partial k_{+1}} \\
\frac{\partial J}{\partial n} &= \frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial n}n - w + (1 - s_n)\beta \frac{\partial J}{\partial n}
\end{aligned}$$

Combining the first-order conditions with the envelope conditions, we obtain, respectively, an expression for capital demand and labor demand

$$\rho = \frac{\frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial k}n + (1 - \delta)s_k\varphi}{1 - (1 - \delta)(1 - s_k)\beta} - \frac{\gamma_k}{\beta p(\theta_k)} \quad (67)$$

$$w = \frac{\frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial n}n - \frac{\gamma_n(r + s_n)}{p(\theta_n)}}{1 - (1 - \delta)(1 - s_k)\beta} \quad (68)$$

The price of capital is implied by the generalized Nash bargaining solution  $\phi_k \frac{\partial J}{\partial k_{+1}} = (1 - \phi_k) \left[ \frac{\partial V}{\partial i} - \frac{\partial V}{\partial u_k} \right] / \lambda$ . Using the above envelope conditions yields

$$\rho = \phi_k \left[ \frac{\frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial k}n + (1 - \delta)s_k\varphi}{1 - (1 - \delta)(1 - s_k)\beta} \right] + (1 - \phi_k)\zeta. \quad (69)$$

Likewise, the wage rate is implied by the generalized Nash bargaining solution  $\phi_n \frac{\partial J}{\partial n} = (1 - \phi_n) \frac{\partial V}{\partial n} / \lambda$ . To derive an explicit expression for  $w$ , it turns out to be convenient to express

$$\begin{aligned}
V_n &= E - U = \lambda w + [1 - s_n(1 - q(\theta_n))]\beta E + s_n(1 - q(\theta_n))\beta U - U \\
&= \lambda w + (1 - s_n)\beta(E - U) + s_n q(\theta_n)\beta(E - U) + \beta U - U \\
&= \frac{\lambda(w - R)}{1 - (1 - s_n)\beta}.
\end{aligned}$$

where  $R \equiv \frac{r}{1+r} \frac{U}{\lambda} - \frac{s_n q(\theta_n)}{1+r} \frac{E-U}{\lambda}$  defines the annuity value of being unemployed; i.e. the worker's reservation wage.<sup>33</sup> Plugging this expression into the Nash bargaining solution yields

$$w = \phi_n \left[ \frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n} [1 - (1 - \delta)(1 - s_k)]k - \frac{\partial w}{\partial n} n \right] + (1 - \phi_n)R. \quad (70)$$

Together with (69), this equation forms a system of non-homogenous differential equations of  $\rho$  and  $w$  in  $k$  and  $n$ . To solve this system, I adapt the derivations laid out in Cahuc, Marque and Wasmer (2007) for the heterogenous labor case. To simplify notation, start by rewriting (69) and (70) as

$$\begin{aligned} \rho &= \tilde{\phi}_k \left[ \frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} \tilde{\delta}k - \frac{\partial w}{\partial k} n \right] + \tilde{\phi}_k (1 - \delta) s_k \varphi + (1 - \phi_k) \zeta \\ w &= \phi_n \left[ \frac{\partial f(k, n)}{\partial n} - \frac{\partial \rho}{\partial n} \tilde{\delta}k - \frac{\partial w}{\partial n} n \right] + (1 - \phi_n)R, \end{aligned}$$

with  $\tilde{\phi}_k = \phi_k / [1 - (1 - \delta)(1 - s_k)\beta]$  and  $\tilde{\delta} = 1 - (1 - \delta)(1 - s_k)$ . Taking the partial derivatives of the two equations with respect to  $n$  and  $k$ , respectively, yields

$$\begin{aligned} \frac{\partial \rho}{\partial n} &= \tilde{\phi}_k \left[ \frac{\partial^2 f(k, n)}{\partial k \partial n} - \frac{\partial^2 \rho}{\partial k \partial n} \tilde{\delta}k - \frac{\partial^2 w}{\partial k \partial n} n - \frac{\partial w}{\partial k} \right] \\ \frac{\partial w}{\partial k} &= \phi_n \left[ \frac{\partial^2 f(k, n)}{\partial k \partial n} - \frac{\partial^2 \rho}{\partial n \partial k} \tilde{\delta}k - \frac{\partial \rho}{\partial n} \tilde{\delta} - \frac{\partial^2 w}{\partial k \partial n} n \right]. \end{aligned}$$

After some rearrangement, the two expressions imply that

$$\frac{\partial \rho}{\partial n} \frac{1 - \tilde{\delta} \tilde{\phi}_k}{\tilde{\phi}_k} = \frac{\partial w}{\partial k} \frac{1 - \phi_n}{\phi_n}.$$

Defining  $\chi_{kn} = \frac{1 - \tilde{\delta} \tilde{\phi}_k}{\tilde{\delta} \tilde{\phi}_k} \frac{\phi_n}{1 - \phi_n}$  and  $\chi_{nk} = 1/\chi_{kn} = \frac{1 - \phi_n}{\phi_n} \frac{\tilde{\delta} \tilde{\phi}_k}{1 - \tilde{\delta} \tilde{\phi}_k}$ , (69) and (70) can thus be reexpressed as

$$\begin{aligned} \rho &= \tilde{\delta} \tilde{\phi}_k \left[ \frac{1}{\tilde{\delta}} \frac{\partial f(k, n)}{\partial k} - \frac{\partial \rho}{\partial k} k - \frac{\partial \rho}{\partial n} \chi_{kn} n \right] + \tilde{\phi}_k (1 - \delta) s_k \varphi + (1 - \phi_k) \zeta \\ w &= \phi_n \left[ \frac{\partial f(k, n)}{\partial n} - \frac{\partial w}{\partial k} \chi_{nk} k - \frac{\partial w}{\partial n} n \right] + (1 - \phi_n)R. \end{aligned}$$

Consider the weighted sums of partial derivatives  $\frac{\partial \rho}{\partial k} k + \frac{\partial \rho}{\partial n} \chi_{kn} n$  and  $\frac{\partial w}{\partial k} \chi_{nk} k + \frac{\partial w}{\partial n} n$  in these expressions. To advance further, introduce auxiliary variables  $\tilde{n}$  and  $\tilde{k}$  such that

$$\frac{\partial \rho(k, n)}{\partial n} \chi_{kn} n = \frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial \tilde{n}} \tilde{n} \quad \text{and} \quad \frac{\partial w(k, n)}{\partial k} \chi_{nk} k = \frac{\partial w(\tilde{k}, n)}{\partial \tilde{k}} \tilde{k},$$

with  $\tilde{\rho}(k, \tilde{n}) = \rho(k, n)$  and  $\partial w(\tilde{k}, n) = w(k, n)$ . Furthermore, define transformed production functions

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<sup>33</sup>Note that in the standard formulation of the labor search model, where separated workers cannot immediately rematch, this reservation wage reduces to  $R = \frac{r}{1+r} U$ , which may look more familiar.

$f(k, n) = g(k, \tilde{n}) = h(\tilde{k}, n)$ . With these auxiliary variables at hand, one can express, respectively

$$\begin{aligned}\frac{\partial \rho}{\partial k} k + \frac{\partial \rho}{\partial n} \chi_{kn} n &= \frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial k} k + \frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial \tilde{n}} \tilde{n} \\ \frac{\partial w}{\partial k} \chi_{nk} k + \frac{\partial w}{\partial n} n &= \frac{\partial w(\tilde{k}, n)}{\partial \tilde{k}} \tilde{k} + \frac{\partial w(\tilde{k}, n)}{\partial n} n.\end{aligned}$$

These expressions can be rewritten in the polar coordinate system. Specifically, let  $k = r \sin \theta$  and  $\tilde{n} = r \cos \theta$ , with  $r$  and  $\theta$  denoting the radial and angular coordinates, respectively. Then

$$\frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial k} k + \frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial \tilde{n}} \tilde{n} = r \frac{\partial \tilde{\rho}(r, \theta)}{\partial r}$$

and thus, the price of capital in (69) can be expressed in the polar coordinate system as

$$\tilde{\rho}(r, \theta) = \tilde{\delta} \tilde{\phi}_k \left[ \frac{g_1(r, \theta)}{\tilde{\delta}} - r \frac{\partial \tilde{\rho}(r, \theta)}{\partial r} \right] + \tilde{\phi}_k (1 - \delta) s_k \varphi + (1 - \phi_k) \zeta,$$

where, for notational simplicity,  $g_1(r, \theta) = \partial g(k(r, \theta), \tilde{n}(r, \theta)) / \partial k$ . This is an independent differential equations of  $\tilde{\rho}$  in  $r$  that can be solved in similar fashion than the differential equation of the basic model. The solution is

$$\tilde{\rho}(r, \theta) = \left[ \int_{\tilde{\delta}}^1 z^{\frac{1-\tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k}} \frac{g_1(rz, \theta)}{\tilde{\delta}} dz + r^{-\frac{1}{\tilde{\delta}\tilde{\phi}_k}} D_\rho(\theta) \right] + \tilde{\phi}_k (1 - \delta) s_k \varphi + (1 - \phi_k) \zeta.$$

The relevant support for the integration derives from the fact that the firm's per-period capital stock ranges from  $(1 - (1 - \delta)(1 - s_k))k = \tilde{\delta}k = \tilde{\delta}r \sin \theta$  (in case of no investment) to  $k = r \sin \theta$  (the actual investment). Under the assumption that  $\lim_{\kappa \rightarrow \tilde{\delta}k} (\kappa - \tilde{\delta}k) \rho(k, n) \rightarrow 0$ , it must be the case that  $D_\rho(\theta)$  is identical to zero. Translating this solution back into the cartesian system in  $k$  and  $\tilde{n}$ , we thus have

$$\tilde{\rho}(k, \tilde{n}) = \int_{\tilde{\delta}}^1 z^{\frac{1-\tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k}} \frac{g_1(kz, \tilde{n}z)}{\tilde{\delta}} dz + \tilde{\phi}_k (1 - \delta) s_k \varphi + (1 - \phi_k) \zeta.$$

A similar transformation yields a solution for the wage rate in (70)

$$\tilde{w}(\tilde{k}, n) = \int_0^1 z^{\frac{1-\phi_n}{\phi_n}} h_2(\tilde{k}z, nz) dz + (1 - \phi_n) R,$$

where  $h_2(\tilde{k}z, nz) = \partial h_2(\tilde{k}z, nz) / \partial(nz)$ . Finally, to determine  $\tilde{n}$  and  $\tilde{k}$ , note that by definition  $\frac{\partial \rho(k, n)}{\partial n} = \frac{\partial \tilde{\rho}(k, \tilde{n})}{\partial \tilde{n}} \frac{\partial \tilde{n}}{\partial n}$  and  $\frac{\partial w(k, n)}{\partial k} = \frac{\partial w(\tilde{k}, n)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k}$ . Hence  $\frac{\partial \tilde{n}}{\partial n} \chi_{kn} n = \tilde{n}$  and  $\frac{\partial \tilde{k}}{\partial k} \chi_{nk} k = \tilde{k}$ . These are two differential equations with solution  $\tilde{n} = n^{1/\chi_{kn}} = n^{\chi_{nk}}$  and  $\tilde{k} = k^{1/\chi_{nk}} = k^{\chi_{kn}}$ .<sup>34</sup> The price of capital and the wage

<sup>34</sup>There are, of course, many other solutions. But we need only one solution and this is presumably the easiest one.

rate become, therefore,

$$\begin{aligned}\rho(k, n) &= \int_{\tilde{\delta}}^1 z^{\frac{1-\tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}}} \frac{f_1(kz, nz^{\chi_{kn}})}{\tilde{\delta}} dz + \tilde{\phi}_k(1-\delta)s_k\varphi + (1-\phi_k)\zeta \\ w(k, n) &= \int_0^1 z^{\frac{1-\phi_n}{\phi_n}} f_2(kz^{\chi_{nk}}, nz) dz + (1-\phi_n)R,\end{aligned}$$

where  $f_1(kz, nz^{\chi_{kn}}) = \partial f(kz, nz^{\chi_{kn}})/\partial(kz)$  and  $f_2(kz^{\chi_{nk}}, nz) = \partial f(kz^{\chi_{nk}}, nz)/\partial(nz)$ .

The solutions for the price of capital and the wage rate imply  $\frac{\partial \rho}{\partial k}k = k \int_{\tilde{\delta}}^1 z^{\frac{1}{\tilde{\delta}\tilde{\phi}_k}} \frac{f_{11}(kz, nz^{\chi_{kn}})}{\tilde{\delta}} dz$  and  $\frac{\partial w}{\partial k}n = n \int_0^1 z^{\frac{1-\phi_n}{\phi_n}} \left(1 + \frac{\tilde{\delta}\tilde{\phi}_k}{1-\tilde{\delta}\tilde{\phi}_k}\right) \frac{f_{12}(kz^{\chi_{nk}}, nz)}{\tilde{\delta}} dz$ . Hence, capital demand in (67) and the price of capital in (69) can be expressed as

$$\rho(k, n) = OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1-\delta)s_k\varphi}{\tilde{\delta}} - \frac{\gamma_k}{\beta p(\theta_k)} \quad (71)$$

$$\rho(k, n) = \phi_k \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1-\delta)s_k\varphi}{\tilde{\delta}} \right] + (1-\phi_k)\zeta, \quad (72)$$

with the overinvestment factor being defined as

$$OI = 1 - \frac{\tilde{\delta}k \int_{\tilde{\delta}}^1 z^{\frac{1}{\tilde{\delta}\tilde{\phi}_k}} \frac{f_{11}(kz, nz^{\chi_{kn}})}{\tilde{\delta}} dz + n \int_0^1 z^{\frac{1-\phi_n}{\phi_n}} \left(1 + \frac{\tilde{\delta}\tilde{\phi}_k}{1-\tilde{\delta}\tilde{\phi}_k}\right) \frac{f_{12}(kz^{\chi_{nk}}, nz)}{\tilde{\delta}} dz}{\frac{\partial f(k, n)/\partial k}{\tilde{\delta}}}.$$

Likewise, labor demand in (68) and the wage rate in (70) become

$$w(k, n) = OE \times \frac{\partial f(k, n)}{\partial n} - \frac{\gamma_n(r + s_n)}{p(\theta_n)} \quad (73)$$

$$w(k, n) = \phi_n \left[ OE \times \frac{\partial f(k, n)}{\partial n} \right] + (1-\phi_n)R, \quad (74)$$

with the overemployment factor being defined as

$$OE = 1 - \frac{\tilde{\delta}k \int_{\tilde{\delta}}^1 z^{\frac{1-\tilde{\delta}\tilde{\phi}_k}{\tilde{\delta}\tilde{\phi}_k}} \left(1 + \frac{\phi_n}{1-\phi_n}\right) f_{12}(kz, nz^{\chi_{kn}}) dz + n \int_0^1 z^{\frac{1}{\phi_n}} f_{22}(kz^{\chi_{nk}}, nz) dz}{\partial f(k, n)/\partial n}.$$

To derive the decentralized equilibrium, combine the optimal capital supply in (66) and the capital demand equation in (71), respectively, with the price of capital in (72)

$$1 - \beta\zeta = q(\theta_k)\phi_k\beta \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1-\delta)s_k\varphi}{\tilde{\delta}} - \zeta \right] \quad (75)$$

$$\gamma_k = p(\theta_k)(1-\phi_k)\beta \left[ OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} + \frac{(1-\delta)s_k\varphi}{\tilde{\delta}} - \zeta \right] \quad (76)$$

Dividing the two equations to eliminate the match surplus in brackets yields the unique decentralized

steady state solution for  $\theta_k \in [0, \infty]$

$$\theta_k^B = \frac{1 - \beta\zeta}{\gamma_k} \frac{1 - \phi_k}{\phi_k}.$$

Given  $\theta_k^B$ , the equilibrium capital-labor ratio  $(k/n)^B$  is pinned down by (76), rewritten as

$$OI \times \frac{\partial f(k, n)/\partial k}{\tilde{\delta}} = \left( \frac{r + \delta + s_k(1 - \delta)}{1 + r} \zeta - s_k(1 - \delta)\varphi \right) + \frac{\gamma_k}{1 - \phi_k} \frac{[r + \delta + s_k(1 - \delta)]}{p(\theta_k)}.$$

In turn, the equilibrium in the labor market is computed by combining the labor demand equation in (73) with the wage equation in (74)

$$(1 - \phi_n) [OE \times \partial f(k, n)/\partial n - R] = \frac{\gamma_n(r + s_n)}{p(\theta_n)}.$$

The reservation wage  $R \equiv \frac{r}{1+r} \frac{U}{\lambda} - \frac{s_n q(\theta_n)}{1+r} \frac{E-U}{\lambda}$  can be made explicit by using the value of unemployment  $\frac{r}{1+r} \frac{U}{\lambda} = b + q(\theta_n)\beta \frac{E-U}{\lambda}$ , the generalized Nash bargaining solution  $\phi_n \frac{\partial J}{\partial n} = (1 - \phi_n) \frac{E-U}{\lambda}$  and the optimal vacancy posting condition  $\gamma_n = p(\theta_n)\beta \frac{\partial J}{\partial n}$

$$R = b + \frac{\phi_n}{1 - \phi_n} \gamma_n \theta_n (1 - s_n).$$

Hence, the above combination of labor demand and wage setting equation becomes

$$OE \times \partial f(k, n)/\partial n = b + \frac{\gamma_n}{1 - \phi_n} \left[ \frac{(r + s_n)}{p(\theta_n)} + \phi_n \theta_n (1 - s_n) \right].$$

This equation pins down the equilibrium labor market tightness  $\theta_n^B$ , provided that  $OE f_n(k, n) - b > 0$ . Finally, the Beveridge curve pins down equilibrium employment

$$n^B = \frac{q(\theta_n)}{s_n + (1 - s_n)q(\theta_n)},$$

which, in turn, determines  $k^B$ . This completes the proof of Proposition 9 and 10. ■