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Risk Aversion and Incentives

Marie-Cécile Fagart
Claude Fluet

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Fagart: Université Paris Descartes
Fluet: Corresponding author: Université du Québec à Montréal and CIRPÉE
fluet.claude-denys@uqam.ca

Abstract:
We consider decision-makers facing a risky wealth prospect. The probability distribution depends on pecuniary effort, e.g., the amount invested in a venture or prevention expenditures to protect against accidental losses. We provide necessary local conditions and sufficient global conditions for greater risk aversion to induce more (or less) investment or to have no effect. We apply our results to incentives in the principal-agent framework when differently risk averse agents face the same monetary incentives.

Keywords: Expected utility, risk aversion, comparative statics, mean utility preserving increase in risk, location independent risk

JEL Classification: D81

1 Introduction

We consider decision-makers facing a risky wealth prospect. The probability distribution depends on pecuniary effort, e.g., the amount invested in a venture or prevention expenditures to protect against accidental losses. The issue is the relation between risk aversion and the decision-maker's effort. This line of inquiry has a long history.[1] We review the literature most relevant for our paper at the end of the introduction. Our contribution is to characterize the necessary "local" or "first-order" conditions for greater risk aversion to induce more (or less) effort and to provide "global" conditions ensuring that the necessary conditions are also sufficient.

To illustrate our approach, consider the so-called LEN model (for *l*inear *e*xponential *n*ormal, see Holmstrom and Milgrom 1987). A decision-maker can invest the amount a in a project yielding a normally distributed gross return Y. The variance is constant and the mean is the concave function $\mu(a)$. The net return is $W \equiv Y - a$. As is well known, all decision-makers with CARA utility functions invest the same amount as would a risk neutral, i.e., they choose a_N maximizing the expected final wealth. When the solution is interior, it satisfies the first-order condition $\mu'(a_N) = 1$. However, the same is also trivially true of all risk averse individuals. Indeed, denoting the utility function by $u(\cdot)$, the marginal expected utility with respect to a can be written as

$$(\mu'(a) - 1)E[u'(W) \mid a].$$

Normally distributed gross returns with a constant variance is thus one example of situations in which risk aversion has no effect on effort. The result is trivial because the final wealth distributions at different investment levels

[1]See for instance the insurance and economics of risk literature, notably Ehrlich and Becker (1972), Dionne and Eeckhoudt (1985), Boyer and Dionne (1989), Briys and Schlesinger (1990), Jullien and al. (1999). More recently Chiu (2005), Eeckhoudt and Gollier (2005), and Meyer and Meyer (2011) discussed the role of prudence in self-protection decisions.

can be ranked on the basis of first-order stochastic dominance.

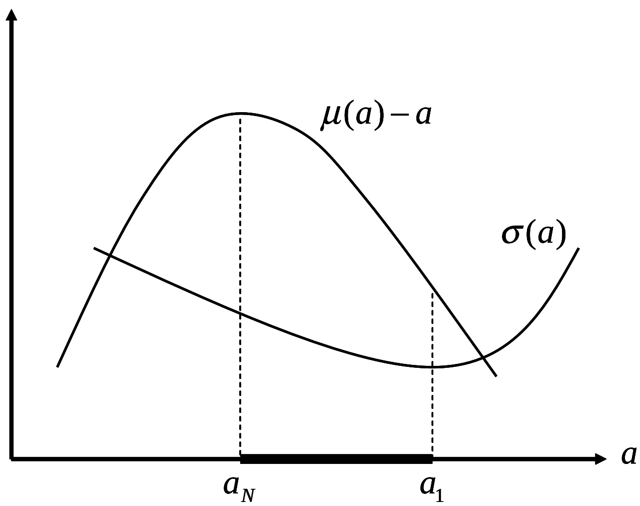

Figure 1. Normally distributed returns

But suppose that the amount invested also affects the standard deviation $\sigma(a)$. The marginal expected utility is now[2]

$$(\mu'(a) - 1)E[u'(W) \mid a] + \sigma'(a)\sigma(a)E[u''(W) \mid a].$$

Obviously, risk neutral decision-makers continue to choose a_N. This is not so for risk averse individuals. For instance, when $\sigma'(a_N) < 0$, the expected utility of a risk averse is strictly increasing in a at a_N. Risk averse decision-makers would therefore be expected to invest more. This must be so when the situation is as represented in Figure 1. With normally distributed returns, the final wealth distribution at a_N second-order stochastically dominates the distributions at all a below a_N. All risk averse decision-makers therefore prefer a_N to any $a < a_N$. Similarly, noting that the variance is minimized at a_1,

[2]The expected utility is $\int u(\mu - a + \sigma z)\phi(z)\, dz$ where $\phi(z)$ is the standard normal density. The marginal expected utility is then $\int u'(\cdot)[\mu' - 1 + \sigma' z]\phi(z)\, dz$. Noting that $z\phi(z) = -\phi'(z)$ and integrating by parts, $\int u'(\cdot) z\phi(z)\, dz = \sigma E[u'' \mid a]$, which yields the second term.

the distribution at a_1 second-order stochastically dominates the distributions at $a > a_1$. Thus, all risk averse decision-makers will choose an action in the open interval (a_N, a_1). The set of second-degree undominated actions is the thick interval in the figure.

One can go a step further. Suppose some $\widehat{a}$ is optimal for a particular risk averse. Because $\sigma'(\widehat{a}) < 0$, one can show that the expected utility of a more risk averse is then increasing at $\widehat{a}$ (and decreasing for a less risk averse). A more risk averse will then invest more than $\widehat{a}$. The result follows because, with $\mu(a)$ concave and $\sigma(a)$ convex as represented in Figure 1, expected utility can be shown to be concave in the amount invested.[3] Altogether, we therefore have a situation where, given the action $\widehat{a}$ chosen by some decision-maker, the choices of more or less averse decision-makers can be predicted solely on the basis of the sign of $\sigma'(\widehat{a})$, i.e., depending on whether risk is "locally" increasing or decreasing with the amount invested.

Suppose, however, that the situation is as represented in Figure 2. Now $\sigma(a)$ reaches a maximum at $a_0 > a_N$ and a minimum at $a_1 > a_0$. A risk neutral continues to choose a_N. Because $\sigma'(a_N) > 0$, a slightly risk averse will choose to invest an amount slightly below a_N. A somewhat more risk averse than the latter individual would invest an even smaller amount. However, a very risk averse decision-maker will invest more than a_N, e.g., he will prefer an action just slightly below a_1. The expected final wealth is then significantly smaller but the risk is sufficiently reduced to make this worthwhile for a very risk averse. In the situation described by Figure 2, expected utility will generally not be concave in a. Indeed, the set of second-degree undominated actions is the union of the two disjoint thick intervals represented in the figure.[4] The example illustrates the following related points. First, when

[3] Differentiate expected utility once more with respect to a and use the same argument as in footnote 2. See section 4 for a more general proof in the case where variables are not normally distributed.

[4] For any a not in this set, there is a' yielding a net return with larger mean and smaller

expected utility cannot be guaranteed to be concave or quasiconcave, the direction of local changes in risk will not allow any firm prediction about the actions chosen by more (or less) risk averse decision decision-makers. Secondly, there are then no general conditions ensuring a monotonic relation between action and risk aversion.

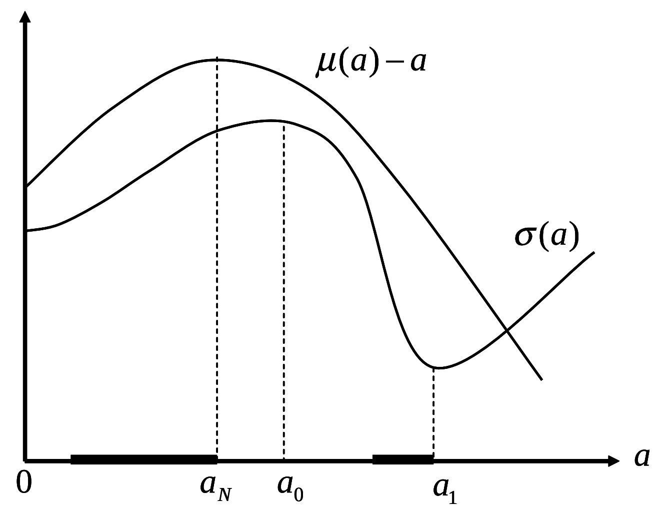

Figure 2. Non convex $\sigma(a)$

This paper provides necessary and sufficient conditions for generalizing to arbitrary distributions the kind of results illustrated in Figure 1. Suppose $\widehat{a}$ is the optimal action of some decision-maker. We first ask whether differently risk averse decision-makers would gain by marginally deviating from $\widehat{a}$? This yields "local" or "first-order" conditions on the distribution of returns for risk aversion to be either locally incentive-neutral, incentive-increasing or decreasing. Next we look at the global maxima of individuals with a utility function differing from that of the reference individual. We show that our "local" conditions for characterizing the relation between risk aversion and action are necessary for the same relation to hold with respect to global maxima. Specifically, for all more risk averse to invest more than $\widehat{a}$, it must be

variance.

the case that expected utility is increasing in a at $\widehat{a}$ for all more risk averse. It follows that, if expected utility is concave in a, the "local" necessary conditions are also sufficient for all more risk averse to invest more. We provide conditions for the decision-makers' problem to be concave. The conditions are shown that to bear no particular relation with how risk varies around $\widehat{a}$. Hence the concavity condition can be combined with "local" conditions ensuring that greater risk aversion is either incentive increasing or decreasing.

In a seminal paper, Diamond and Stiglitz (1974) introduced the notion of mean utility preserving increase (or decrease) in risk. They use the concept to analyze the effect of risk aversion on behavior in various contexts. Applied to a problem such as ours, their analysis is purely in terms of local comparative statics results: given the action chosen by some reference decision-maker, a marginally more risk averse individual gains by investing marginally more (or less). This sides step the difficulties raised in the example illustrated in Figure 2. By contrast, we obtain the same condition as Diamond and Stiglitz' as a *necessary* condition for *all* more risk averse to invest more. Next we provide conditions ensuring that the local characterization of changes in risk is *sufficient* to predict global optima. An important feature, as mentioned above, is that the global conditions have no bearing on local changes in risk. The analogy with the example of Figure 1 is that the conditions $\mu'' \leq 0$ and $\sigma'' \geq 0$ imply nothing concerning the sign of $\sigma'(\widehat{a})$.

Another important contribution is Jewitt's (1989) notion of location independent risk. The notion is important because it can be used to characterize the "dispersion" of gross returns independently of the amount invested (or of the individual's initial wealth). Applied to our problem, one can show that a sufficient condition for a monotonic relation between risk aversion and action is that location independent risk be monotonic in the amount invested. However, the assumption of overall monotonicity would not fit the example illustrated by Figure 1 where risk is first decreasing then increasing. Both the notions of mean utility preserving increase in risk and location independent risk nevertheless play an important role in our analysis as a characterization of local changes in risk.

Section 2 sets out the decision problem and reviews notions of risk and dispersion. Section 3 derives the necessary condition. Section 4 compares global optima and presents examples. Section 5 applies the results to incentives in the principal-agent model when effort has a money cost. Section 6 concludes. All proofs are in the Appendix.

2 Set-up

Individuals invest $a \in [0, \overline{a}]$ in a prospect yielding the gross return y, so that final wealth is $w = y - a$. Returns are realized according to the distribution $G(y \mid a)$, where G is twice-continuously differentiable with density denoted by g. For all a, the support is contained in $\mathbb{Y} \equiv [\underline{y}, \overline{y}]$, where the bounds need not be finite. Utility functions are strictly increasing in final wealth, concave, and twice-continuously differentiable. We denote by $\mathbb{U}$ the set of all such utility functions defined over $[\underline{w}, \overline{w}] \equiv [\underline{y} - \overline{a}, \overline{y}]$.

Individuals will be referred to by their utility function in $\mathbb{U}$. We write utility functions as lower-case letters and use the corresponding upper case for the expected utility. For decision-maker u, the expected utility from investing a is

$$U(a) \equiv \int_{\underline{y}}^{\overline{y}} u\,(y-a)\,g(y \mid a)\,dy. \tag{1}$$

We consider distributions of returns such that (1) exists.

Our purpose is to compare the choices of decision-makers who differ in risk aversion, including risk neutrality as a limiting case. Individual v is more risk averse than individual u if v is a nondecreasing concave transformation of u or equivalently

$$\frac{-v''(w)}{v'(w)} \geq \frac{-u''(w)}{u'(w)} \text{ for all } w. \tag{2}$$

Note that differences in risk aversion may be due to differences in wealth, e.g., one could write $u(y-a) \equiv \psi(w_0^u + y - a)$ where w_0^u is individual u's initial wealth.

Various notions of risk and stochastic orders will prove useful. Diamond and Stiglitz' (1974) mean utility preserving increase in risk, hereafter DS-riskiness, is defined with respect to a reference individual. Denote the distribution of final wealth by $H(w \mid a)$, so that $H(w \mid a) \equiv G(w + a \mid a)$. Distributions can be thought of as being indexed by a. By definition, a_2 is DS-riskier than a_1 with respect to the utility u if individual u is indifferent between a_1 and a_2 while all individuals more risk averse than u prefer a_1. This property is equivalent to

$$\int_{\underline{w}}^{w} u'(\eta)H(\eta \mid a_1)\, d\eta \leq \int_{\underline{w}}^{w} u'(\eta)H(\eta \mid a_2)\, d\eta \text{ for all } w, \tag{3}$$

$$\int_{\underline{w}}^{\overline{w}} u'(\eta)H(\eta \mid a_1)\, d\eta = \int_{\underline{w}}^{\overline{w}} u'(\eta)H(\eta \mid a_2)\, d\eta. \tag{4}$$

When the reference individual is risk neutral so that u' is constant, (3) and (4) imply that a_2 is a mean preserving spread of a_1. With u' constant, condition (3) on its own means that a_1 second-degree stochastically dominates a_2. All risk averse individuals then prefer a_1 (at least weakly) and so does a risk neutral.

Jewitt' (1989) location independent risk, hereafter J-dispersion, does not rely on a reference individual. It ranks distributions independently of horizontal shifts in the distributions.[5] The notion generalizes the idea that a normally distributed variable is more risky than another if it has a larger variance, irrespective of the means. J-dispersion can therefore be applied directly to gross returns. The distribution of returns a_2 has more J-dispersion

[5] On these notions, see also Landsberger and Meilijson (1994) and Chateauneuf et al. (2004).

than a_1 if

$$\int_{\underline{y}}^{Q(p,a_1)} G(y \mid a_1)\, dy \leq \int_{\underline{y}}^{Q(p,a_2)} G(y \mid a_2)\, dy \text{ for all } p \in (0,1), \tag{5}$$

where $Q(p,a)$ denotes the quantile function associated with $G(y \mid a)$, i.e., $Q(p,a)$ is the p-percentile gross return. Because our cumulative distributions are strictly increasing in y, the quantile is simply the inverse function defined by

$$G(Q(p,a) \mid a) \equiv p \text{ for all } p \in (0,1). \tag{6}$$

Using the change of variable $y = Q(p,a)$ and integrating by parts, condition (5) can be rewritten as

$$\int_0^p \eta Q_p(\eta, a_1)\, d\eta \leq \int_0^p \eta Q_p(\eta, a_2)\, d\eta \text{ for all } p \in (0,1). \tag{7}$$

We will also sometimes refer to the following stronger condition. The distribution of returns a_2 is more dispersed than a_1 in the sense of Bickel and Lehmann (1979), hereafter BL-dispersion, when $Q(p,a_2) - Q(p,a_1)$ is non decreasing in p, i.e., when

$$Q_p(p,a_1) \leq Q_p(p,a_2) \text{ for all } p \in (0,1). \tag{8}$$

Obviously, (8) implies 7.

Finally, it will also be useful to express the condition of second-degree stochastic dominance in terms of quantiles (see Levy 1992). The quantile function of final wealth is $Q(p,a) - a$. The distribution a_1 second-degree dominates a_2 if

$$\int_0^p (Q(\eta,a_1) - a_1)\, d\eta \geq \int_0^p (Q(\eta,a_2) - a_2)\, d\eta \text{ for all } p \in (0,1). \tag{9}$$

3 Necessary conditions

Decision-makers choose a to maximize expected utility. An interior maximum for individual u satisfies the first-order condition

$$U'(a) = -\int_{\underline{y}}^{\overline{y}} u'(y-a)\left(G_a(y \mid a) + g(y \mid a)\right) dy = 0. \tag{10}$$

We assume throughout that maxima are interior. We first analyze necessary conditions for all more risk averse individuals to invest more (or less) than some reference decision-maker.

Local incentives. The necessary conditions are in terms of "local incentives", by which me mean what happens in the neighborhood of some action.

Definition 1 *Let $U'(a_u) = 0$ for decision-maker u. Risk aversion is (i) incentive-neutral at a_u if $V'(a_u) = 0$ for all v; (ii) incentive-increasing (resp. decreasing) at a_u if $V'(a_u) \geq 0$ (resp. ≤ 0) for all v more risk averse than u, with strict inequalities for some v.*

Incentive-neutrality means that, if a_u is a stationary point for individual u, then the same is true for all decision-makers irrespective of their degree of risk aversion. Risk aversion is incentive-increasing when all more risk averse decision-makers weakly prefer to marginally increase their investment above a_u and some strictly so.

Proposition 1 *Let $U'(a_u) = 0$ for decision-maker u. Risk aversion is*
(i) incentive-neutral at a_u if and only if

$$G_a(y \mid a_u) + g(y \mid a_u) = 0, \text{ for all } y \in \mathbb{Y}; \tag{11}$$

(ii) incentive-increasing at a_u if and only if, with strict inequalities over some interval,

$$\int_{\underline{y}}^{y} u'(\eta - a_u)\left(G_a(\eta \mid a_u) + g(\eta \mid a_u)\right) d\eta \leq 0 \text{ for all } y \in \mathbb{Y}, \tag{12}$$

and is incentive-decreasing when the reversed inequalities hold.

When a_u is in fact optimal for individual u, condition (11) is necessary for all individuals to also want to choose a_u. It turns out that the local condition (12) is itself necessary for the optimal choice of all more risk averse individuals to be above a_u.

Proposition 2 *Let a_u and a_v be optimal for the individuals u and v respectively. If $a_v = a_u$ for all v, then (11) holds. If $a_v \geq a_u$ for all v more risk averse than u, then (12) holds; if $a_v \leq a_u$ for all v more risk averse than u, the reverse inequalities hold.*

The second part of the proposition rules out the possibility that a decision-maker would prefer to marginally decrease his investment below a_u while his global maximum is in fact above a_u. Slightly different individuals can be expected to make only slightly different choices, i.e., there will be some whose best action is arbitrarily close to a_u. For all decision-makers more risk averse than u to invest more than a_u therefore requires the expected utilities of slightly different decision-makers to be non-decreasing at a_u. The proposition shows that the same must in fact be true for all more risk averse decision-makers.[6]

Local changes in risk and dispersion. We now interpret our necessary conditions in terms of risk and dispersion. Proposition 2 imposes conditions on the distribution of final wealth, which we denoted $H(w \mid a)$. Consider first the case where all individuals in $\mathbb{U}$ optimally choose a_u. It must then be

[6] We focused on *more* risk averse decision-makers when (12) or the reverse inequalities hold. Because the condition is necessary and sufficient for risk aversion to be locally increasing (or decreasing), the implications for *less* risk averse decision-makers are perfectly symmetrical.

that a_u second-degree stochastically dominates all a. This requires

$$\int_{\underline{w}}^{w} H(\eta \mid a_u)\, d\eta \leq \int_{\underline{w}}^{w} H(\eta \mid a)\, d\eta \text{ for all } w \text{ and } a.$$

The expression on the right-hand side is therefore minimized at a_u, for all w. Because the minimum is interior, the necessary first-order condition is

$$\int_{\underline{w}}^{w} H_a(\eta \mid a)\, d\eta = 0 \text{ for all } w.$$

Differentiating with respect to w then yields $H_a(w \mid a_u) = 0$ for all w. Condition (11) follows by noting that

$$H_a(y - a \mid a) \equiv G_a(y \mid a) + g(y \mid a).$$

Thus, (11) is the first-order condition for second-degree stochastic dominance at a_u.

It may be remarked that (11) is also the first-order condition for first-degree stochastic dominance at a_u. The requirement is then $H(w \mid a_u) \leq H(w \mid a)$ for all a and w, which again implies $H_a(w \mid a_u) = 0$ for all w. Obviously, if there exists a first-degree dominant action, it will also be second-degree dominant. In the case of normally distributed returns represented in Figure 1, a_N is second-degree dominant when $\sigma(a)$ also reaches its minimum at a_N (i.e., when $a_1 = a_N$). However, unless $\sigma(a)$ is constant, there is then no first-degree dominant action.

Consider now condition (12) when a_u is optimal for u. In terms of the distribution of final wealth at a_u and $a_u + \varepsilon$, where ε is positive but negligible,

$$\int_{\underline{w}}^{w} u'(\eta) \left(H(\eta \mid a_u + \varepsilon) - H(\eta \mid a_u)\right)\, d\eta \simeq \varepsilon \int_{-\infty}^{w} u'(\eta) H_a(\eta \mid a_u)\, d\eta.$$

Because decision-maker u's expected utility is maximized at a_u, a small change has only second-order effects, i.e., the right-hand side vanishes at

$w = \overline{w}$. Given condition (12), however, the right-hand side is non positive for all $w \leq \overline{w}$ when $\varepsilon > 0$. The condition therefore states that a marginally larger investment level is DS-less risky than a_u with respect to u. Hence, it is preferred by decision-makers more risk averse than u.

We now turn to the notions of dispersion introduced in section 2. These can be defined in the neighborhood of a given action, noting that for ε arbitrarily small $Q_p(p, a + \varepsilon) - Q_p(p, a) \simeq \varepsilon Q_{ap}(p, a)$.

Definition 2 *BL-dispersion is locally decreasing in a if $Q_{ap}(p, a) \leq 0$ for all p. J-dispersion is locally decreasing in a if $\int_0^p \eta Q_{ap}(\eta, a)\, d\eta \leq 0$ for all p. Dispersion is stationary if the equalities hold for all p.*

From (6), it is easily seen that $Q_p(p, a) = g(Q(p, a) \mid a)^{-1}$ and

$$Q_a(p, a) = -\frac{G_a(Q(p, a) \mid a)}{g(Q(p, a) \mid a)} \text{ for } p \in (0, 1).$$

Condition (11) is therefore equivalent to $Q_a(p, a_u) = 1$ for all p. It follows that $Q_{ap}(p, a_u) = 0$ for all p, i.e., condition (11) implies that dispersion is stationary at a_u.

BL-dispersion decreasing at a_u implies (12). To see this, write expected utility in terms of the quantile function,

$$U(a) = \int_0^1 u(Q(p, a) - a)\, dp. \tag{13}$$

Expected utility is stationary at a_u when

$$U'(a_u) = \int_0^1 u'\,(y - a_u)\,(Q_a(p, a_u) - 1)\; dp = 0. \tag{14}$$

Similarly, (12) can be rewritten as

$$\int_0^p u'\,(y - a_u)\,(Q_a(\eta, a_u) - 1)\; d\eta \geq 0 \text{ for } p \in (0, 1). \tag{15}$$

Given condition (14), $Q_a(p, a_u) - 1$ cannot be always positive or always negative. When $Q_{ap}(p, a_u) \leq 0$ for all p, the expression is nonincreasing in p

and must therefore change from positive to negative. Hence, BL-dispersion decreasing at a_u implies (15). We show that J-dispersion decreasing at a_u also implies (15).

Corollary 1 *Given $U'(a_u) = 0$, condition (11) is equivalent to dispersion being stationary at a_u; J-dispersion decreasing at a_u implies condition (12).*

Discussion. When u optimally chooses a_u, condition (12) is necessary for all individuals more risk averse than u to invest more than a_u. However, the condition is not sufficient by itself to predict global optima, unless expected utility can be guaranteed to be concave or pseudoconcave in the amount invested.[7] The next section will provide sufficient conditions.

Even when one abstracts from these issues, it should be emphasized that a condition such as (12) does not yield an overall monotonic relation between optimal actions and risk aversion. Specifically, it may well be that some other individual v, who cannot be ranked in terms of risk aversion with respect to u, also optimally chooses a_u. For individual v, the equivalent of condition (12) may not hold, even though it holds for u. Hence, because the condition is necessary to allow predictions, there will be some individuals more risk averse than v who will invest less than v and some who will invest more.

We briefly discuss the implications of overall monotonicity between action and risk aversion. Let $\mathbb{A} \equiv \{a \mid a \in \arg\max_{a'} U(a'),\ u \in \mathbb{U}\}$. When $\mathbb{A}$ is not a singleton, it is the set of actions which cannot be ranked by second-degree stochastic dominance. In other words, any action in this set is some individual's optimal choice. An example is the interval $[a_N, a_1)$ in Figure 1 for the case of normally distributed returns.

Corollary 2 *Let a_u and a_v denote the optimal actions for u and v respectively. If $\mathbb{A}$ is convex and $a_v \geq a_u$ for all u, $v \in \mathbb{U}$ with v more risk averse*

[7] A continuously differentiable quasiconcave function is pseudoconcave if a vanishing derivative implies a maximum (rather than an inflection point or a "flat section").

than u (alternatively $a_v \leq a_u$), then $U(a)$ is quasiconcave over $\mathbb{A}$ for all $u \in \mathbb{U}$.

Figure 4 provides an illustration. The set $\mathbb{A}$ is the interval from a_l to a_h. According to the corollary, each decision-makers' expected utility is single-peaked over the set of relevant actions. This follows from Proposition 2. In the figure, v is more risk averse than u and $a_v > a_u$. Hence, v's expected utility is non-decreasing at a_u. But the same is also true for all $a < a_v$ because any such a is chosen by some less risk averse than v. A similar argument shows that v's expected utility is everywhere non-increasing to the right of a_v (see the Appendix for the details).

Overall monotonicity requires strong conditions on G. In the next section, we do not impose such strong conditions. Our focus is the use of the local conditions of Proposition 1 for predicting behavior.

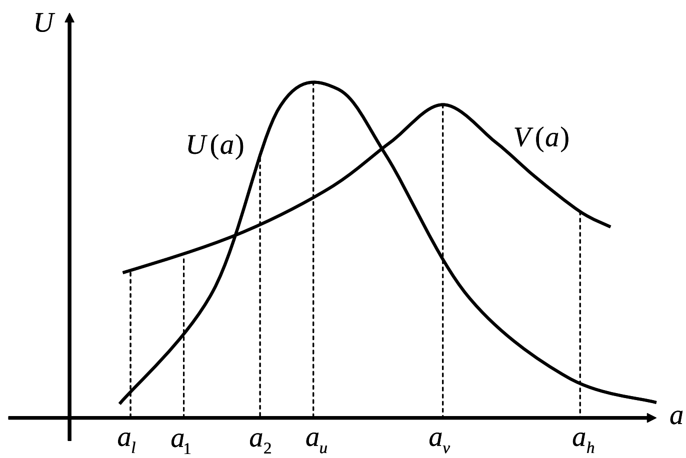

Figure 3. Monotonicity and quasiconcavity

4 Local incentives as sufficient conditions

Given the action optimally chosen by individual u, the conditions of Proposition 1 are sufficient for predicting the choices of more (or less) averse decision-makers if expected utility is concave in effort for all decision-makers. Recalling the quantile expression for expected utility in (13), a straightforward sufficient condition is the concavity in a of the quantile function $Q(p, a)$, equivalently $G(y \mid a)$ quasiconvex in y and a. A weaker condition is Concavity of the Cumulative Quantile (CCQ) as defined in Fagart and Fluet (2013). They show that the condition is sufficient for expected utility to be concave in a for all risk averse (or risk neutral) decision-makers.

Definition 3 *The distribution $G(y \mid a)$ satisfies CCQ if $\int_0^p Q(\eta, a)\, d\eta$ is concave in a for all p, equivalently if $\int_{\underline{y}}^y G(\eta \mid a)\, d\eta$ is convex in (y, a).*

When the gross return distribution satisfies CCQ, condition (11) implies that all decision-makers also choose a_u; condition (12) implies that all more risk averse decision-makers invest more than a_u. We discuss two examples illustrating how quantile concavity or the weaker CCQ combine with local changes in dispersion.

Multiplicative-additive risk model. Let the gross return be a linear function of some random state, i.e., $y = \mu(a) + \sigma(a)z$ where $\sigma(a) > 0$ and z is the realization of the random variable Z which does not depend on a. This generalizes the case of normally distributed returns discussed in the introduction. Without loss of generality, we can take Z to have zero mean and unit variance, hence $\mu(a)$ and $\sigma(a)$ are respectively the mean and the standard deviation of returns.

Let $Q_Z(p)$ denote the quantile function of Z. The zero mean condition is $\int_0^1 Q_Z(p)\, dp = 0$. The gross return quantile is

$$Q(p, a) = \mu(a) + \sigma(a)Q_Z(p).$$

We assume $\mu''(a) \leq 0$ and $\sigma''(a) \geq 0$. These conditions guarantee that the distribution of returns satisfy CCQ: for all p,

$$\int_0^p Q_{aa}(\eta, a)\, d\eta = p\mu''(a) + \sigma''(a) \int_0^p Q_Z(\eta)\, d\eta \leq 0.$$

The integral on the right-hand side is negative because it has the same sign as the mean of the distribution truncated from the right.

Let a_u be individual u's optimal action. Recalling the quantile formulation of second-degree stochastic dominance in (9), observe that

$$\int_0^p \left(Q_a(\eta, a_u) - 1\right)\, d\eta = p\left(\mu'(a_u) - 1\right) + \sigma'(a_u) \int_0^p Q_Z(\eta)\, d\eta$$

cannot be positive (or negative) for all p. If it were, a marginal variation from a_u would second-degree dominate, contradicting the statement that a_u is optimal.

There are therefore three possibilities. The first is $\mu'(a_u) - 1 = \sigma'(a_u) = 0$, implying that expected final wealth is maximized and that its variance is minimized. Noting that $Q_{ap}(p, a) = \sigma'(a) Q'_Z(p)$, this possibility corresponds to the case where condition (11) holds and dispersion is stationary at a_u. The action a_u is then second-degree dominant and all decision-makers also choose a_u.

The two other possibilities are either $\mu'(a_u) > 1$ together with $\sigma'(a_u) > 0$ or $\mu'(a_u) < 1$ together with $\sigma'(a_u) < 0$. When $\sigma'(a_u) < 0$, BL-dispersion is decreasing at a_u. Individual u invests more than the amount that would maximize expected wealth because this allows him to "purchase" less dispersion. Condition (12) then holds, implying that more risk averse decision-makers will invest even more than u. When $\sigma'(a_u) > 0$, BL-dispersion is increasing at a_u. Individual u invests less than the the amount maximizing expected wealth because this would come at the cost of too much dispersion. More risk averse individuals will then invest even less.

Stochastic production function. Consider now a general stochastic production function with a as input, $y = \varphi(a, z)$ where $\varphi_a > 0$, $\varphi_{aa} < 0$ and $\varphi_z > 0$, i.e., large values of z correspond to more favorable states of Nature. The gross return quantile is then $Q(p, a) = \varphi(a, Q_Z(p))$. Concavity in a follows from the concavity of the production function, hence expected utility is concave in a. Note that $Q_{ap}(p, a) = \varphi_{az}(a, Q_Z(p)) Q'_Z(p)$.

Suppose decision-maker u optimally chooses a_u. If $\varphi_{az}(a_u, z) < 0$ in all states of Nature, BL-dispersion is decreasing at a_u. Individuals more risk averse than u invest more because the marginal product of investment is relatively larger in unfavorable states of Nature.[8] More risk averse individuals are more willing than u to trade off a smaller net wealth in favorable states against a larger net wealth in unfavorable ones. Conversely, if $\varphi_{az}(a_u, z) > 0$ in all states, BL-dispersion is increasing at a_u and more risk averse individuals invest less than u.

A more general case is to allow $\varphi_{az}(a_u, z)$ to change sign. For instance, it may be that the marginal product φ_a is below average when Nature is either very harsh or very generous; conversely that it is above average in extreme conditions, whether favorable or unfavorable. J-dispersion is decreasing at a_u if

$$\int_0^p \eta \varphi_{az}(a_u, Q_Z(\eta)) Q_Z'(\eta)\, d\eta \leq 0 \text{ for all } p.$$

Integrating by parts, this can be rewritten as

$$\varphi_a(a_u, Q_Z(p)) \leq \frac{1}{p} \int_0^p \varphi_a(a_u, Q_Z(\eta))\, d\eta \text{ for all } p. \tag{16}$$

The expression on the right-hand side is the average marginal product below the p-percentile. Condition (16) is equivalent to

$$\frac{d}{dp}\left(\frac{1}{p}\int_0^p \varphi_a(a_u, Q_Z(\eta))\, d\eta\right) \leq 0 \text{ for all } p.$$

In words, J-dispersion is decreasing at a_u when the marginal product is on average larger in unfavorable states. Individuals more risk averse than u then invest more.

First versus second-degree stochastic dominance. When dispersion is stationary at some given action and the distribution of returns satisfies

[8] This is essentially the case discussed in Jullien et al. (1999). Their Propositions 5 and 6 are both equivalent to the condition that BL-dispersion is decreasing at all a. Observe that $\varphi_{az}(a_u, z) < 0$ for all z only means that BL-dispersion is locally decreasing at a_u.

CQC, all risk averse or risk neutral decision-makers optimally choose that action. In other words, the local condition (11) is then both necessary and sufficient for the action to be second-degree dominant. In some cases the action may also be first-degree dominant and will be so if a first-degree dominant action exists.

In section 3, we remarked that the same condition (11) was also necessary for first-degree stochastic dominance at the action considered. Are there conditions that would allow us to infer first-degree stochastic dominance when (11) holds? For this purpose, we consider an alternative characterization of decreasing returns to investment. The condition will be referred to Isoprobability Convexity of the Distribution Function (ICDF).

Definition 4 *The distribution of returns satisfies ICDF if* $G_a(Q(p,a) \mid a)$ *is nondecreasing in* a *for all* p *and* a*.*

The condition evokes the well known Convexity of the Distribution Function Condition (CDFC) defined by $G_a(y \mid a)$ nondecreasing in a for all y. However, ICDF looks at convexity for given probability levels. When a larger a increases returns in the sense of first degree stochastic dominance, $-G_a(y \mid a)$ is the marginal productivity in terms of reducing the probability of outcomes worse than y. Similarly, $-G_a(Q(p,a) \mid a)$ is the p-level marginal productivity. ICDF imposes that p-levels marginal returns be decreasing. The condition is easily shown to be equivalent to $G_a(y \mid a)/g(y \mid a)$ nondecreasing in a.

Corollary 3 *Suppose* $U'(a_u) = 0$ *and dispersion is stationary at* a_u*. If the distribution of returns satisfies CCQ,* a_u *is second-degree stochastically dominant. If it satisfies ICDF,* a_u *is first-degree dominant.*

When dispersion is stationary at a_u, ICDF ensures that the net return quantile, $Q(p,a) - a$, is nondecreasing in a for $a < a_u$ and nonincreasing otherwise; that is, the net return quantile is quasiconcave and reaches a maximum at a_u for all p. Hence a_u is first-degree dominant.[9] There is some overlap between CCQ and ICDF. For instance, G_a always nonpositive, ICDF and BL-dispersion everywhere nonincreasing can be shown to imply CCQ.[10]

5 An application to incentive schemes

We apply the preceding results to incentives in the principal-agent framework with moral hazard. Effort is pecuniary, i.e., an unverifiable monetary expenditure. The principal faces many different agents and is constrained to use the same performance scheme with all of them. The issue is how agents with different risk preferences react to the same monetary incentives.

The wage or payment depends on the signal X, e.g., the agent's output or a purely informative performance index. The cdf is denoted $\Psi(x \mid a)$, the corresponding density is $\psi(x \mid a)$ with a support $\mathbb{X} \equiv [\underline{x}, \overline{x}]$ that is invariant with respect to effort. We make the usual assumptions that Ψ is twice continuously differentiable and satisfies MLRP with ψ_a/ψ strictly increasing in x so that $\Psi_a < 0$. Given an increasing wage scheme, the marginal gross return of effort is positive in the sense of first degree stochastic dominance.

We show that the effect of risk aversion depends on the curvature of the payment scheme, where curvature is in terms of an appropriate transformation of the signal. Loosely speaking, when the wage scheme is "concave", risk aversion will be incentive increasing; it will be incentive decreasing when the scheme is "convex".

For some effort level $\widehat{a}$, consider the transformed signal Z defined by

$$z = \varphi(x) \equiv -\int_{x_c}^{x} \frac{\psi(\eta \mid \widehat{a})}{\Psi_a(\eta \mid \widehat{a})}\, d\eta,\ x \in \mathbb{X}, \tag{17}$$

[9]The reverse condition, $G_a(Q(p,a) \mid a)$ nonincreasing in effort, would imply that the expected utility reaches a global minimum when dispersion is stationary at a_u.

[10]MLRP (i.e., g_a/g increasing in y), ICDF and Jewitt-dispersion everywhere decreasing implies CCQ.

where x_c is some arbitrary threshold of the original signal. Denote the support of the transformed signal by $\mathbb{Z} \equiv [\underline{z}, \overline{z}]$. Because φ is a strictly increasing function, Z also satisfies MLRP and delivers the same information as X with respect to a. Hence, any wage scheme defined with respect to the original signal can be replicated as a scheme defined with respect to the transformed signal. We henceforth focus on the latter and represent a wage scheme by a function $y = y(z)$.

As a first observation, under a scheme of the form $y(z) = z$, risk aversion is incentive-neutral at $\widehat{a}$. To see this, observe that the resulting random wage is distributed according to the cdf $F(z \mid a) = \Psi(\varphi^{-1}(z) \mid a)$ with density $f(z \mid a) = \psi(\varphi^{-1}(z) \mid a) d\varphi^{-1}(z)/dz$. Noting that

$$\frac{d\varphi^{-1}(z)}{dz} = -\frac{\Psi_a(\varphi^{-1}(z) \mid \widehat{a})}{\psi(\varphi^{-1}(z) \mid \widehat{a})},$$

it follows that $F_a(z \mid \widehat{a}) + f(z \mid \widehat{a}) = 0$ for all z, i.e., condition (11) of Proposition 1 holds at $\widehat{a}$. The same would be true of any linear scheme $y(z) = z + k$ where k is an arbitrary constant. We now look at the effects of non linearity.

Corollary 4 *Under the wage scheme $y(z)$, BL-dispersion is decreasing at $\widehat{a}$ if and only if $y''(z) \leq 0$ for all z. J-dispersion is decreasing at $\widehat{a}$ if and only if $E[y'(Z) \mid Z \leq z, \widehat{a}]$ is decreasing in z for all z.*

Decreasing BL-dispersion at $\widehat{a}$ is equivalent to the concavity of the payment scheme compared to the linear benchmark. Under concave schemes, a marginal increase in effort above $\widehat{a}$ reduces the riskiness of the wage distribution; the opposite occurs with convex schemes. The interpretation with respect to J-dispersion is similar. J-dispersion is decreasing at $\widehat{a}$ when the scheme is "concave on average", i.e., the slope of the wage function is on average larger at the bottom the distribution. The intuition is that whether risk aversion is incentive increasing or decreasing depends on the location of incentives with respect to realizations of the signal.

We now provide a characterization in terms of condition (12) of Proposition 1.

Corollary 5 *Suppose* $U'(\widehat{a}) = 0$ *under the scheme* $y(z)$ *and define* $f_u(z) \equiv u'(y(z) - \widehat{a})f(z \mid \widehat{a})$. *Then*

$$\int_{\underline{z}}^{\overline{z}} (y'(z) - 1) f_u(z)\, dz = 0 \tag{18}$$

and risk aversion is incentive increasing at $\widehat{a}$ *if and only if*

$$\int_{\underline{z}}^{z} (y'(\eta) - 1) f_u(\eta)\, d\eta \geq 0 \text{ for all } z. \tag{19}$$

The function $f_u(z)$ can be interpreted as a density because utility functions are arbitrary up to an increasing linear transformation. Condition (19) describes a scheme tilted towards penalizing the agent for bad performance rather than rewarding him for good performance.

When u is risk neutral, $f_u(z) \equiv f(z \mid \widehat{a})$. The conditions (18) and (19) then reduce to

$$E[y'(Z) \mid \widehat{a}] = 1, \tag{20}$$

$$E[y'(Z) \mid Z \leq z, \widehat{a}] \geq 1 \text{ for all } z. \tag{21}$$

A risk neutral agent is motivated only by the overall expectation of the slope. He is indifferent to whether the slope is on average greater at the bottom or at the top of the distribution. This is not so for a risk averse agent. Given (21), the expected utility of a risk averse is increasing at $\widehat{a}$: exerting more effort yields a mean preserving contraction in the agent's random wage. The opposite obtains when the inequalities are reversed. When agent u is himself risk averse, condition (19) together with (18) imply that the density f_u puts greater weight on bad outcomes compared to good ones. A more risk averse agent would put even greater weight on bad outcomes.

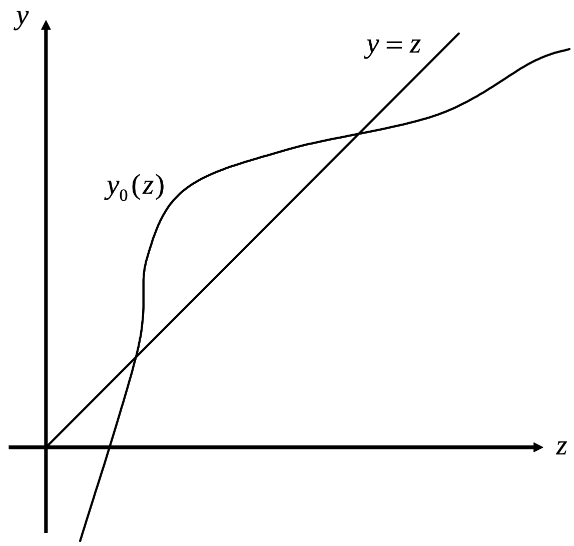

Figure 4. A penalty oriented scheme

Figure 4 provides an illustration. In the figure, $y'(z) - 1$ is first positive then negative as z increases. The single-crossing property implies that, if (20) holds, then (21) must also hold. Similarly, if (18) holds, then so must (19).

Because a risk averse agent is sensitive to the location of the incentives, it is possible to induce effort with a scheme that is not very steep overall, provided it has sufficient "power" in the lower tail. The more risk averse the agent, the more the slope in the lower tail will matter compared to that in the upper tail. Therefore, depending on the location of incentives, wages that provide risk neutral agents with relatively weak incentives may provide strong incentives for risk averse ones. Conversely, a reward-oriented scheme that provides good incentives for a risk neutral may work poorly with risk averse individuals.

6 Concluding Remarks

We derived necessary local conditions on the probability distributions for greater risk aversion to be incentive neutral, incentive-increasing or decreasing. Combined with conditions ensuring that the decision problems are concave, the local conditions are sufficient for comparing the choices of differentially risk averse decision-makers. Because the conditions are necessary, it follows that, when they do not hold, choices cannot be compared on the basis of risk aversion alone. It may be possible, nevertheless, to make predictions for a restricted class of utility functions, e.g., functions exhibiting positive prudence or downside risk aversion. A first step in this direction was recently taken by Eeckhoudt and Gollier (2005) and Chiu (2005). An interesting extension of the present paper would be to look for necessary and sufficient conditions for comparing the choices of decision-makers ordered in terms of downside risk aversion as defined in Keenan and Snow (2009).

Appendix

We first prove a preliminary result.

Lemma 1 *Let γ be a piecewise continuous integrable function on $\mathbb{Y} = [\underline{y}, \overline{y}]$. The inequality $L(\mu) = \int_{\mathbb{Y}} \gamma(y)\mu(y)\, dy \geq 0$ holds for all continuous non negative functions μ if and only if $\gamma(y) \geq 0$ for all $y \in \mathbb{Y}$.*

Proof. Sufficiency is obvious. To show necessity, choose $y_1 < y_2$ in $\mathbb{Y}$ and ε such that $0 < \varepsilon < \dfrac{y_2 - y_1}{2}$. Define

$$\begin{aligned} \mu_\varepsilon(y) &= 0 \text{ if } y \notin [y_1, y_2], \\ &= \frac{y - y_1}{\varepsilon} \text{ if } y \in [y_1, y_1 + \varepsilon), \\ &= 1 \text{ if } y \in [y_1 + \varepsilon, y_2 - \varepsilon), \\ &= \frac{y_2 - y}{\varepsilon} \text{ if } y \in [y_2 - \varepsilon, y_2]. \end{aligned}$$

Observe that $\mu_\varepsilon(y)$ is non negative and continuous. Moreover,

$$\lim_{\varepsilon \to 0} L(\mu_\varepsilon) = \int\limits_{y_1}^{y_2} \gamma(y)\, dy \geq 0,$$

where the inequality follows by continuity. As a consequence, there exists no interval (y_1, y_2) such that $\gamma(t) < 0$ for all $t \in (y_1, y_2)$. Because $\gamma(y)$ is continuous on appropriate intervals, $\gamma(y) \geq 0$ for all $y \in \mathbb{Y}$.

Proof of Proposition 1.

Sufficiency. Differentiating the decision-maker's expected utility with respect to effort and integrating by parts yields

$$\begin{aligned} U'(a) &= \int\limits_{\mathbb{Y}} u(y-a) g_a(y \mid a)\, dy - \int\limits_{\mathbb{Y}} u'(y-a) g(y \mid a)\, dy \\ &= -\int\limits_{\mathbb{Y}} u'(y-a)(G_a(y \mid a) + g(y \mid a))\, dy. \end{aligned}$$

For $a = a_u$, the sufficiency of (11) in part (i) is then obvious. For (ii), define

$$\Delta_u(y) \equiv -\int_{\underline{y}}^{y} u'(\eta - a_u)\left(G_a(\eta \mid a_u) + g(\eta \mid a_u)\right) d\eta, \text{ for } y \in \mathbb{Y}. \qquad (22)$$

Then $U'(a_u) = \Delta_u(\overline{y}) = 0$ and condition (12) requires $\Delta_u(y) \geq 0$ for all y. For decision-maker v,

$$V'(a) = -\int\limits_{\mathbb{Y}} \frac{v'(y-a_u)}{u'(y-a_u)} \Delta_u'(y)\, dy.$$

Integrating by parts and using $\Delta_u(\overline{y}) = 0$ implies

$$V'(a_u) = -\int\limits_{\mathbb{Y}} \frac{\partial}{\partial y}\left(\frac{v'(y-a_u)}{u'(y-a_u)}\right) \Delta_u(y)\, dy.$$

When v is more risk averse than u, $\partial(v'/u')/\partial y \leq 0$ so that (12) implies $V'(a_u) \geq 0$.

Necessity. Using the notations in lemma 1, define

$$\begin{aligned} \gamma(y) &= \Delta_u(y), \\ \mu(y) &= -\frac{\partial}{\partial y}\left(\frac{v'(y-a_u)}{u'(y-a_u)}\right), \end{aligned}$$

so that $V'(a_u) = L(\mu)$. Applying lemma 1, $L(\mu) \geq 0$ for all non negative and continuous $\mu(y)$ implies that $\gamma(y)$ must be non negative. This proves the necessity of (12) in part (ii). To prove necessity in part (i), we show that (11) must hold if $V'(a_u) = 0$ for all more risk averse decision-makers. The equality is equivalent to $V'(a_u) \geq 0$ and $V'(a_u) \leq 0$. By lemma 1, the first inequality implies that $\gamma(y)$ is non negative, the second that $-\gamma(y)$ is non negative. Therefore $\gamma(y) = 0$ for all y. Differentiating with respect to y then yields (11). QED

Proof of Proposition 2. Suppose an interior maximum for u; in case of multiplicity, let a_u be the largest one. Then $U'(a_u) = 0$ and $U(a_u) > U(a)$ for all $a > a_u$. Suppose all decision- makers more risk averse than u invest more and there exists one, say v, such that $V'(a_u) < 0$. As v's expected utility is strictly decreasing at a_u, there exists $\widehat{a} > a_u$ such that $V(a_u) > V(a)$ for all $a \in [a_u, \widehat{a}]$. Define

$$\begin{aligned} \beta(a) &= \frac{V(a) - V(a_u)}{U(a_u) - U(a)} \text{ for all } a \geq \widehat{a}, \\ \beta &= \max_{a \in [\widehat{a}, \overline{a}]} \beta(a). \end{aligned}$$

Observe that $\beta(a)$ is continuous in $[\widehat{a}, \overline{a}]$, so β exists. Moreover, $\beta > 0$. Indeed, if $\beta \leq 0$, then $V(a_u) \geq V(a)$ for all $a \in [a_u, \overline{a}]$ and v will prefer a_u to any larger a. As $V'(a_u) < 0$, decision-maker v's best choice would then be below a_u, contradicting the statement that all more risk averse choose a larger a.

Consider now a decision-maker with $\widetilde{u} = v + \beta u$. This decision-maker is more averse than u and $\widetilde{U}'(a_u) < 0$. Moreover, $\widetilde{U}(a_u) \geq \widetilde{U}(a)$ for all $a \geq a_u$. Indeed,

$$\widetilde{U}(a_u) - \widetilde{U}(a) = V(a_u) - V(a) + \beta\left(U(a_u) - U(a)\right).$$

When $a \in [a_u, \widehat{a}]$, $\widetilde{U}(a_u) - \widetilde{U}(a) > 0$ because $V(a_u) \geq V(a)$ and $U(a_u) > U(a)$. When $a \geq \widehat{a}$,

$$\widetilde{U}(a_u) - \widetilde{U}(a) \geq V(a_u) - V(a) + \beta(a)\left(U(a_u) - U(a)\right) = 0,$$

where the last equality follows from the definition of $\beta(a)$. As $\widetilde{U}'(a_u) < 0$ and $\widetilde{U}(a_u) \geq \widetilde{U}(a)$ for all $a \geq a_u$, decision-maker $\widetilde{u}$'s best choice is below a_u, yielding a contradiction. Thus, $V'(a_u) \geq 0$ is necessary for all v more risk averse than u.

Proof of Corollary 1. The equivalence between (11) and stationary BL-dispersion is obvious. We show that locally decreasing J-dispersion is sufficient for (12). Let $\Delta_u(y)$ be defined as in (22). Define

$$\widehat{\Delta}_u(p) \equiv \Delta_u(Q(p, a_u)) = \int_0^p u'(Q(\eta, a_u) - a_u)(Q_a(\eta, a_u) - 1)\, d\eta$$

and observe that $\Delta_u(y) \geq 0$ for all y is equivalent to $\widehat{\Delta}_u(p) \geq 0$ for all $p \in (0,1)$. Observe that Δ_u is zero at both the lower and upper bounds of $\mathbb{Y}$, i.e., $\widehat{\Delta}_u(0) = \widehat{\Delta}_u(1) = 0$. It follows that $\widehat{\Delta}_u(p)$ is everywhere nonnegative if it is nonnegative at interior local extrema, i.e., at values of p satisfying $\widehat{\Delta}'_u(p) = 0$. By continuity, such extrema necessarily exist when $\widehat{\Delta}_u$ is not constant over $(0,1)$. Thus

$$\widehat{\Delta}_u(p) \geq 0 \text{ for all } p \in \mathbb{P}^* \Rightarrow \widehat{\Delta}_u(p) \geq 0, \text{ all } p \in (0,1)$$

where $\mathbb{P}^* \equiv \{p \in (0,1) \mid Q_a(p, a_u) = 1\}$ is the set of stationary points. It therefore suffices to show that decreasing J-dispersion implies that $\widehat{\Delta}_u(p)$ is nonnegative at stationary points. Integrating $\widehat{\Delta}_u(p)$ by parts then yields

$$\widehat{\Delta}_u(p) = u'(Q(p, a_u) - a_u)\widehat{\Delta}(p) - \int_{\underline{y}}^{y} u''(Q(\eta, a_u) - a_u)Q_p(\eta, a_u)\widehat{\Delta}(\eta)\, d\eta,$$

where

$$\widehat{\Delta}(p) \equiv \int_0^p (Q_a(\eta, a_u) - 1)\, d\eta.$$

We next show that decreasing J-dispersion implies that $\widehat{\Delta}(p) \geq 0$ for all $p \leq \sup \mathbb{P}^*$. This will imply that $\widehat{\Delta}_u(p) \geq 0$ for all $p \in \mathbb{P}^*$. Note that

$$\begin{aligned} \widehat{\Delta}(p) &= p\,(Q_a(G(y, a_u), a_u) - 1) - \int_0^p \eta Q_{ap}(\eta, a_u)\, d\eta \\ &= -\int_0^p \eta Q_{ap}(\eta, a_u)\, d\eta \geq 0 \text{ when } p \in \mathbb{P}^*. \end{aligned}$$

The latter is non negative under decreasing J-dispersion. Because $\mathbb{P}^*$ is also the set of stationary points of $\widehat{\Delta}(p)$, it follows that $\widehat{\Delta}_u(p) \geq 0$ for $p \leq \sup \mathbb{P}^*$. QED

Proof of Corollary 2. To complete the argument in text, note that the implications for *less* risk averse decision-makers (recall footnote 5). Thus, if $a_u \leq a_v$ for u less risk averse than v, then $U'(a_v) \leq 0$. Hence $U(a)$ must be nonincreasing at all $a \geq a_u$. QED

Proof of Corollary 3. The argument in the text shows that a_u is second-degree dominant under CCQ. For any individual v, $V'(a)$ can be rewritten as

$$V'(a) = -\int_{\mathbb{Y}} v'(y - a) \left(\frac{G_a(y \mid a)}{g(y \mid a)} + 1 \right) g(y \mid a)\, dy.$$

Under ICDF, $G_a(y \mid a)/g(y \mid a)$ is nonincreasing in a, so that $V'(a) \geq \Omega_v(a)$ if and only if $a \leq a_u$, where

$$\Omega_v(a) \equiv -\int_{\mathbb{Y}} v'(y - a) \left(\frac{G_a(y \mid a_u)}{g(y \mid a_u)} + 1 \right) g(y \mid a)\, dy.$$

When $G_a(y \mid a_u) + g(y \mid a_u) = 0$ for all y, hence $\Omega_v(a) = 0$ for all a. It follows $V(a)$ reaches a maximum at a_u. QED

Proof of Corollary 4. Denote by $Q^Z(p,a)$ the quantile for the random variable Z. For the wage $y(z)$, the quantile is then $Q(p,a) = y(Q^Z(p,a))$. Therefore

$$Q_{ap}(p,a) = y''(Q^Z(p,a))Q_a^Z(p,a)Q_p^Z(p,a) + y'(Q^Z(p,a))Q_{ap}^Z(p,a).$$

By construction of the signal Z, $Q_{ap}^Z(p,\widehat{a}) = 0$ for all p and $Q_a^Z(p,\widehat{a}) = 1$. We thus have:

$$Q_{ap}(G(z \mid \widehat{a}),\widehat{a}) \leq 0 \Leftrightarrow y''(z) \leq 0.$$

For J-dispersion, a similar argument shows that

$$\begin{aligned}\int_0^p pQ_{ap}(\eta,\widehat{a})\,d\eta &= \int_0^p p\left(y''(Q^Z(\eta,\widehat{a}))Q_p^Z(\eta,\widehat{a})\right)\,d\eta \\ &= py'(Q^Z(p,\widehat{a})) - \int_0^p y'(Q^Z(\eta,\widehat{a}))\,d\eta.\end{aligned}$$

Hence

$$\int_0^{F(z|\widehat{a})} pQ_{ap}(p,\widehat{a})\,dp \leq 0 \Leftrightarrow E[y'(Z) \mid Z \leq z,\widehat{a}] \geq y'(z).$$

The latter is easily shown to be equivalent to $E[y'(Z) \mid Z \leq z,\widehat{a}]$ nonincreasing in z as

$$E[y'(Z) \mid Z \leq z,\widehat{a}] = \frac{\int_0^{F(z|\widehat{a})} y'(Q^Z(\eta,\widehat{a}))\,d\eta}{F(z \mid \widehat{a})}.$$

QED.

Proof of Corollary 5. Under the scheme $y(z)$, $G(y(z) \mid a) = F(z \mid a)$ so

$$\begin{aligned}\int_{\underline{y}}^{y_0} u'(\eta - a)(G_a(\eta \mid \widehat{a}) + g(\eta \mid \widehat{a}))\,d\eta &= \int_{\underline{z}}^{y^{-1}(y_0)} u'(y(z)-a)(y'(z)F_a(\eta \mid \widehat{a}) + f(z \mid \widehat{a}))\,dz \\ &= -\int_{\underline{z}}^{y^{-1}(y_0)} u'(y(z)-a)(y'(z)-1)f(z \mid \widehat{a}))\,dz,\end{aligned}$$

noting that by construction $F_a(z \mid \widehat{a}) = -f(z \mid \widehat{a})$ for all z. Therefore (12) is equivalent to (19). QED

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